Why Prices Don't Respond Sooner to a Prospective Sovereign Debt Crisis.*

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Abstract

Since 2008 actions have been taken in Europe and elsewhere that increase the cost of short-selling sovereign debt. We show that such actions can have a profound effect on the timing and magnitude of price responses to bad news in periods leading up to a sovereign default. When financial markets are frictionless, prices drop instantly in response to bad news even if the prospect of a crisis is very remote. Imposing costs on short-selling disrupts this dynamic. Government bond prices exhibit no response to bad news when the prospects are remote. Instead price declines only occur immediately prior to a sovereign default and then in a nonlinear way.

Keywords: sovereign debt crisis; bond prices. leverage; heterogenous beliefs.

JEL Classification numbers: E62, H60.

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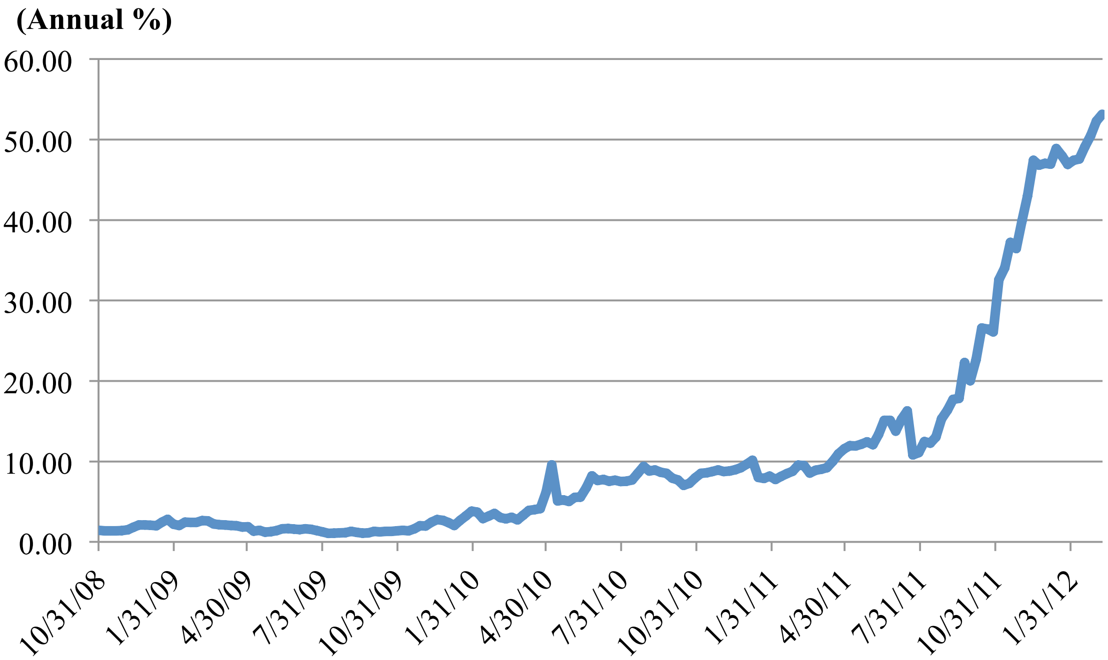
Figure 1: Greece 10yr Government Bond Yield Premium over Germany.



1 Introduction

Data on government debt yields from historial defaults have the property that the biggest movements in bond yields occur shortly before the default event. Figure 1, which reports the yield premium on 10 year Greek debt relative to Germany illustrates a typical pattern. The figure has the shape of a hockey stick facing backwards. Increases in bond premia are relatively low between 2008 and August 2011 and then increase sharply between November of 2011 and March 9 of 2013 when a credit event is declared on Greek sovereign debt credit default swaps (CDS). The largest increases in the Greek yield premium occur between November 2011 and March 9 2013. What is striking is that these large movements were preceded by a long string of bad news reports that date back to 2009. For instance, Fitch downgraded Greek debt from A- to BBB+ in December of 2009. Eurostat announced that their Greek public debt statistics were not reliable on January 12, 2010 and Greece requested its first bailout from the IMF and EU on April 23, 2010.

Greece is not unique. Paluszynski (2015) points out that other peripheral countries in the EU also experienced large declines in GDP and a worsening in their trade balance in 2008 but that yields on their sovereign debt did not begin to respond until two years later. Nieto-Parra

(2009) using data from 13 sovereign debt crises, finds that investment banks start charging significantly higher fees to underwrite sovereign debt one to three years in advance of a default but that sovereign bond yields don't begin to rise until shortly before the crisis.

These empirical observations are puzzling because bond prices are determined by participants beliefs about future payoffs. Thus, one would expect that bond prices would react in a strong way to news suggesting that the risk of a sovereign debt crisis has increased. Yet, bond prices appear to be lagging other indicators available to investors instead.

One explanation for these observations is that the negative content of news occurring shortly before a crisis is particularly large. This can happen, for instance, if a sovereign chooses to strategically delay releasing bad news about the risk of a sovereign default to the market. Braun, Mukerjee and Runkle (1996) and Paluszynski (2015) develop theories where a sovereign has superior information that a default is likely and yet is able to successfully delay releasing this information without it impinging on bond yields.

In this paper we provide an alternative explanation for the backwards facing hockey stick pattern in bond yields. Our explanation assumes no informational asymmetries and instead relies on a particular type of financial friction. It is not uncommon for policy makers to attribute sharp unfavorable changes in the price of government liabilities to the actions of short-sellers. Short-selling activities are subject to special regulations in many countries (see Angel (2004) for a description of regulatory restrictions on short-selling in the U.S., Europe and Asia.). These regulations make it more costly for investors to take short-positions as compared to long positions. It is also not uncommon for sovereigns to increase the costs of short-selling when the price of government obligations including debt and/or currency falls. Germany banned naked short-sales of sovereign CDSs in 2010. In November of 2012 this ban was extended to the entire Euro area. Governments also take actions to increase the costs of shot-sellers when their currencies are threatened. Some of the more extreme measures include splitting onshore and off-shore currency markets (Spain in 1992 and Thailand in 1997), imposing capital controls (Malaysia in 1998), or undertaking large interventions in equity markets (Hongkong in 1998).[1]

We show that short-selling costs can account for the reverse hockey-stick pattern in sovereign bond yields by developing an equilibrium model of sovereign debt markets. Individuals in our model have heterogenous beliefs about the probability of a sovereign debt crisis. Agents who are optimistic can borrow to purchase government debt, and agents who

[1]In August 1998 the Hongkong Monetary Authority purchased domestic stocks amounting to about 7% of the Hongkong Stock Exchange's total market capitalization and 30% of its free float in an effort to fend off short-sellers. See the discussion in Corsetti, Pesenti and Roubini (2001) for more details.

are pessimistic can short government debt. Both agents are subject to collateral constraints that restrict the sizes of their positions. Those taking long-positions hold government bonds as collateral and those taking short positions hold cash as collateral. The leverage rates are determined endogenously as in Geanakoplos (2003, 2010). Trade occurs in multiple periods and default is explicit.[2]

We find that modeling costly short-selling has a big impact on the timing and magnitude of bond price movements to the same sequence of bad news shocks. When short-selling costs are zero one period holding returns drop sharply in response to bad news about the possibility of a future debt crisis, even if the event is very distant. However, when short-sales are costly some potential short-sellers of government debt find it too costly to trade on their beliefs and choose to remain on the sidelines as bad news starts to arrive. Only very optimistic agents participate in sovereign bond markets and one-period holding returns on bonds don't react to the first bad news. As the default event approaches there is a burst in participation and one-period holding returns fall sharply. The resulting pattern of bond yields has the same backwards pointing hockey stick pattern shown in Figure 1.

We also investigate the welfare properties of costly short-selling. Welfare comparisons are more subtle in our model because there is no objective truth and agents heterogenous beliefs about the prospects of a sovereign-default are all equally valid. We find that the imposition of costs on short-selling can be justified by a Rawlsian welfare criterion. Agents in our model are risk neutral and the fraction of agents that go bankrupt is lower when short-selling is costly.

Our model of costly short-selling is related to models with heterogeneous beliefs and financial frictions considered by Geanakoplos (2003, 2010). He investigates the role of ruling out short-sales on asset pricing. Our model extends the work of Geanakoplos by allowing for both leveraged short and long-sales and differs in other respects due to our interest in sovereign default.

The combination of heterogenous beliefs and an exogenous ban on short sales has also been used by Harrison and Kreps (1978), Scheinkman and Xiong (2003) and Hong and Sraer (2011) to account for bubbly phenomena in asset prices. For instance Scheinkman and Xiong (2003) show that agents are willing to purchase an asset even when it exceeds their evaluation of its fundamental value because they expect to be able to sell it in the future at a higher price. We consider versions of our model with multiple period bonds. However, the subjective evaluation of cash flows for optimistic agents who purchase these bonds exceeds the equilibrium price. It follows that this type of bubble does not arise in our model.

[2]In Braun and Nakajima (2014) we also report results for an economy where a sovereign default is implicitly engineered by inflation. This scenario is more plausible for countries such as the U.S. and Japan.

Our research is also complementary to previous research by Bi (2011) and Bi, Leeper and Leith (2012). These papers also produce nonlinear movements in bond rates leading up to a sovereign default in representative agent dynamic general equilibrium models. The source of the nonlinearity in bond rates in their setup is nonlinearities in the objective probability of default. We also generate nonlinearities in the dynamics of bond yields. In our model the nonlinearities are jointly determined by the initial distribution of beliefs, the market structure and the resulting patterns of trade in the bond market. The principal message of our analysis is that the micro-structure of the bond market is important. Financial frictions and asymmetries in the cost of short-selling government debt creates nonlinearities and magnifies any nonlinearities that might arise in frictionless financial markets.

The remainder of the paper is organized as follows. Section 2 describes the 1-period model, Section 3.2 describes the T-period model, and Section 4 contains our concluding remarks.

2 The One-period model

We consider the situation of a small open economy in a currency union such as Greece. In particular, we assume that the demand for loans is allowed to differ from the domestic supply of loans, that money supply is determined outside of the economy and that the aggregate price level is exogenous. Our objective is to show how costs on selling government bonds short affect the dynamics of government bond prices and the pattern of trade along a history resulting in a sovereign default. In order to isolate these effects it will be helpful to also consider an Arrow-Debreu (AD) market structure where financial frictions are absent. Some of the features of the economy are common across the two market structures so we describe them first.

The one-period model has two instants of time that are indexed by $t = 0, 1$. There are two states of nature in period 1, U and D, that are distinguished by whether the government defaults on its debt. Default occurs in state D. We let s_t denote the state of nature in period t, where $s_0 = 0$ and $s_1 \in S \equiv \{U, D\}$.

Prior to time zero all individuals agree that the probability of a sovereign default is zero. At the beginning of time zero before any trade takes place bad news arrives. Agents interpret the bad news in different ways and this induces a non-degenerate distribution of beliefs $h \in [0, 1]$ over the probability of a default in period 1.

Government: Government policy is exogenous. The government starts off with $\bar{B} \geq 0$ nominal liabilities to the private sector that mature in period 1. Default occurs in period 1 in state D: $\alpha(D) < 1$. In state U we have $\alpha(U) = 1$. The government raises revenue to pay off any remaining debt by taxing agents' period 1 endowments. Let $T(s_1)$ denote the total real amount of taxes in state $s_1 \in S$ of period 1. Then the flow budget constraint of the government in period 1 is

$$\alpha(s_1)\frac{\bar{B}}{P_1} = T(s_1), \qquad s_1 \in S \tag{1}$$

where $\alpha(s_1)$ is the fraction of debt repaid in state s_1 and P_1 is the price level in period 1. The price level evolves according to $P_1 = \Pi P_0$ where $\Pi = 1 + \pi$ is the gross inflation rate. Thus a government policy is given by $(\alpha(s_1), T(s_1)) \in (0, 1]^2 \times \mathbb{R}^2_{++}$.

Individuals: We index agents $h \in [0, 1]$ by their subjective probability of state U. Specifically, we assume that agents of type h believe that $s_1 = U$ with probability h. Thus, agents with high h assign less probability to the debt crisis event. Let $f(h)$ denote the density of type h agents, where $f(h) \geq 0$ for all h and $\int_0^1 f(h)\, dh = 1$.

Agents have linear preferences of the form

$$u_h = c_0 + hc(U) + (1 - h)c(D). \tag{2}$$

We assume that consumption is nonnegative in all dates and states throughout the analysis.

Agents also have different endowments of the consumption good and government bonds. Let $y_t > 0$, $t = 0, 1$, denote the aggregate endowment of the consumption good in period t. We will assume throughout that it does not depend on whether default occurs in period 1, $y_1(U) = y_1(D) = y_1$. An agent of type-h is endowed with $e(h)y_t$ units of the good in period t. All agents endowments are nonnegative: $e(h)$ satisfies $e(h) \geq 0$ for all $h \in [0, 1]$. Individual endowments of the consumption good are linked to the aggregate endowment in the following way

$$\int_0^1 g(h)y_t\, dh = y_t, \qquad \text{for all } t,$$

where $g(h) \equiv f(h)e(h)$ is the density of the distribution of agents and $G(h)$ is the associated cumulative distribution function. Agents are also endowed with $e(h)\bar{B}$ units of government bonds in period 0 and have access to a risk-free storage technology that offers a gross real rate of return R.

Taxes are assumed to be proportionate to the endowment of the consumption good in period 1. An individual with endowment $e(h)y_1$ pays $e(h)T(s_1)$ in taxes to the government. These assumptions insure that each agent has sufficient resources to pay taxes in period 1 and facilitates comparing bond prices in the two market settings that we consider.

2.1 Leverage Market Structure

in the *leverage market structure* trade in Arrow claims is ruled out but agents can take leveraged long and short positions on government debt in private loan markets.

It follows from our assumption of risk neutrality that optimistic agents, who believe that the rate of return on government bonds is greater than the borrowing rate, will want to borrow as much as possible and use the proceeds to purchase government bonds. Their total positions are limited by the requirement that they post government bonds as collateral in order to obtain a loan. How much can an agent borrow with one unit of government bonds as collateral? One way to proceed would be to impose an exogenous ad hoc constraint as in e.g. Kiyotaki and Moore (1997). We pursue an alternative avenue that determines the collateral constraint endogenously. Geanakoplos (2003, 2010) posits a broad array of loan/default contracts and determines which ones trade in equilibrium. Applying this approach to our model yields a "no-default constraint," that requires that the amount of repayments not exceed the value of the collateral in any state. We simplify the ensuing exposition of the model by directly imposing the no-default constraint.

Since there is no default on loans, loans are risk-free. Thus the interest rate on loans is equal to R in equilibrium (as long as the storage technology is used). Consider an agent who borrows ϕ_0 and purchases government bonds b_0 in period 0. She must repay $R\phi_0$ in period 1. The no-default constraint requires that $R\phi_0 \leq \frac{b_0}{P(s_1)}$, for all $s_1 \in S$.

Agents who believe that the rate of return on government bonds is lower than the borrowing rate, will want to borrow as much government debt as they can acquire, sell it today, purchase it back tomorrow at the anticipated lower price and return the government debt to the lender. In practice, short-sellers have to post collateral, we assume that the collateral is money which is not subject to default in our setting but does lose value over time when there is inflation.[3] Short-sellers are also subject to a no default condition.

Let k_0^h denote the amount of safe storage by agent h, m_0^h the amount of money held by agent h, b_0^h the amount of government debt held by agent h and ϕ_0^h the amount of loans obtained by agent h. Given these definitions the budget constraints in period 0 and 1 for agent h are

$$c_0^h + k_0^h + \frac{1}{P_0}m_0^h + \frac{q_0}{P_0}b_0^h = e(h)\left(\frac{q_0}{P_0}\bar{B} + y_0\right) + \phi_0^h - \chi_0^h, \tag{3}$$

$$c^h(s_1) = \frac{\alpha(s_1)}{P_1}b_0^h + R(k_0^h - \phi_0^h) + \frac{1}{P_1}m_0^h + e(h)\big[y_1 - T(s_1)\big], \qquad s_1 \in S. \tag{4}$$

[3]Posting cash is equivalent to posting claims to safe storage when $\Pi R = 1$.

Here q_0 is the price of government bonds, and χ_0^h is a proportionate fee on short sales of government bonds

$$\chi_0^h = \chi \max\left\{0, -q_0 \frac{b_0^h}{P_0}\right\}. \tag{5}$$

The collateral constraints are

$$\frac{\alpha(s_1)}{P_1} b_0^h - R\phi_0^h \geq 0, \qquad s_1 \in S, \tag{6}$$

$$\frac{\alpha(s_1)}{P_1} b_0^h + \frac{m_0^h}{P_1} \geq 0, \qquad s_1 \in S. \tag{7}$$

Constraint (6) imposes the restriction that loans received to acquire bonds do not exceed the value of bonds in any state and constraint (7) imposes the condition that agents who wish to short-bonds hold sufficient collateral in the form of money to deliver bonds in any state. Even though we are directly imposing these constraints, as we noted above, they can be derived as endogenous constraints by proceeding in the same fashion as Geanakoplos (2003, 2010).

Definition 1 (Leverage Competitive Equilibrium)**.** Given a distribution of endowments (G, y_0, y_1, $\bar{B}$), a sequence of prices (P_0, P_1) and government policy $(\alpha(s_1), T(s_1))$, a leverage competitive equilibrium consists of an allocation $\{c_0^h,\ [c^h(s_1)]_{s_1\in S},\ k_0^h,\ m_0^h,\ b_0^h,\ \phi_0^h\}_{h\in[0,1]}$ and a price of bonds q_0, such that (i) each agent $h \in [0,1]$, $\{c_0^h,\ [c^h(s_1)]_{s_1\in S},\ k_0^h,\ m_0^h,\ b_0^h,\ \phi_0^h\}$ solves her utility maximization problem; (ii) the government budget constraint (1) is satisfied; and (iii) the market for government bonds clears

$$\bar{B} = \int_0^1 b_0^h f(h)\, dh. \tag{8}$$

This definition omits any reference to a market clearing condition for money. We are implicitly assuming that the demand of money by short-sellers is small relative to the supply of money in the currency union.

Characterization of the Leverage Equilibrium: Agents hold one of three portfolios in a leverage equilibrium. The most optimistic agents borrow and use the proceeds to purchase government bonds. The most pessimistic agents short-sell government bonds and hold cash. The remaining agents choose not to participate in the government bond market and are indifferent between lending to optimists and saving using the safe storage technology. The range of individuals who hold each of these portfolios is determined by the following indifference relations

$$R = \bar{h}_0 \left(\frac{q_0}{P_0} - \frac{1}{R} \frac{\alpha(D)}{P_1} \right)^{-1} \left(\frac{\alpha(U)}{P_1} - \frac{\alpha(D)}{P_1} \right), \tag{9}$$

$$R = (1 - \underline{h}_0) \left(\frac{\alpha(U)}{P_0} - (1-\chi) \frac{q_0}{P_0} \right)^{-1} \left(\frac{\alpha(U)}{P_1} - \frac{\alpha(D)}{P_1} \right). \tag{10}$$

Equation (9) is the indifference relationship for a marginal purchaser of government bonds. For individual $\bar{h}_0$ the return from safe storage is equal to the return from borrowing at the rate R and using the proceeds to purchase government bonds. Equation (10) is the corresponding relationship for individual $\underline{h}_0$ who is indifferent between safe storage and holding money as collateral in order to make short sales of government bonds.

Given these definitions we can now describe the optimal portfolios of each type of investor. Agents $h > \bar{h}_0$ will take leveraged long positions in government debt. Their optimal portfolio is

$$\begin{aligned} k_0^h &= m_0^h = 0 \\ \phi_0^h &= \frac{\alpha(D)}{RP_1} b_0^h \\ b_0^h &= \frac{P_1}{\Pi q_0 - \alpha(D)/R} e(h) \left(y_0 + \frac{q_0}{P_0} \bar{B} \right). \end{aligned} \tag{11}$$

where $\frac{P_1}{\Pi q_0 - \alpha(D)/R}$ is the amount of leverage available to them.

For agents $h \in [\underline{h}_0, \bar{h}_0]$ the optimal portfolio is

$$\begin{aligned} b_0^h &= m_0^h = 0 \\ k_0^h - \phi_0^h &= e(h) \left(y_0 + \frac{q_0}{P_0} \bar{B} \right) \end{aligned} \tag{12}$$

and the remaining agents $h \in [0, \underline{h}_0)$ are short-sellers of government bonds and with portfolio

$$\begin{aligned} k_0^h - \phi_0^h &= 0 \\ m_0^h &= -\alpha(U) b_0^h \\ b_0^h &= -\frac{P_0}{\alpha(U) - (1-\chi) q_0} e(h) \left(y_0 + \frac{q_0}{P_0} \bar{B} \right). \end{aligned} \tag{13}$$

where $\frac{P_0}{\alpha(U) - (1-\chi) q_0}$ is the amount of leverage available to short-sellers.

Using equations (11), (12) and (13) we can now express the government bonds market clearing condition as

$$\frac{\bar{B}}{P_0} = \left\{ \int_{\bar{h}_0}^{1} \frac{1}{q_0 - \frac{\alpha(D)}{\Pi R}} g(h)\, dh - \int_0^{\underline{h}_0} \frac{1}{\alpha(U) - (1-\chi) q_0} g(h)\, dh \right\} \left(y_0 + \frac{q_0}{P_0} \bar{B} \right). \tag{14}$$

Equation (14) in conjunction with (9) and (10) allows us to solve for $\bar{h}_0$, $\underline{h}_0$ and q_0. Specifically, using equations (9) and (10), write the marginal agents, $\bar{h}_0$ and $\underline{h}_0$, as functions of the bond price q_0: $\bar{h}_0(q_0)$ and $\underline{h}_0(q_0)$. Then equation (14) is expressed as

$$S(q_0) = D(q_0), \tag{15}$$

where supply, $S(q_0)$, and demand, $D(q_0)$, for government bonds in period 0 is given by

$$S(q_0) \equiv \frac{1}{P_0}\frac{\bar{B}}{y_0} + G\big[\underline{h}_0(q_0)\big]\frac{1+\frac{q_0}{P_0}\frac{\bar{B}}{y_0}}{\alpha(U)-(1-\chi)q_0}, \tag{16}$$

$$D(q_0) \equiv \Big\{1 - G\big[\bar{h}_0(q_0)\big]\Big\}\frac{1+\frac{q_0}{P_0}\frac{\bar{B}}{y_0}}{q_0-\frac{\alpha(D)}{\Pi R}}. \tag{17}$$

Inspection of equations (15)-(17) reveals two noteworthy properties about the determination of government bond prices. First, bond prices only depend on the debt-output ratio. Consequently, in the analysis that follows we hold fixed the level of output and vary the amount of government debt. Second, the pattern of endowments and beliefs only enter via G. In other words, the aggregate endowment for each type: $e(h)f(h)$ matters for the determination of the price of bonds but not the individual distributions of endowments and beliefs. This is an example of the well known anonymity or equal treatment property of competitive equilibrium. In the analysis that follows we thus consider shifts in the total endowment for each type (G) but remain agnostic about $f(h)$ and $e(h)$.

Lemma 1. *The leverage equilibrium exists and is unique.*

Proof. Note that

$$\frac{d}{dq_0}\left(\frac{y_0+\frac{q_0}{P_0}\bar{B}}{\alpha(U)-(1-\chi)q_0}\right) > 0, \quad \frac{d}{dq_0}\left(\frac{y_0+\frac{q_0}{P_0}\bar{B}}{q_0-\frac{\alpha(D)}{\Pi R}}\right) < 0,$$
$$\frac{d\underline{h}_0(q_0)}{dq_0} > 0, \quad \frac{d\bar{h}_0(q_0)}{dq_0} > 0.$$

It follows that the supply and demand curve for government bonds have the standard slopes:

$$S'(q_0) > 0, \quad \text{and} \quad D'(q_0) < 0.$$

Note also that $\lim_{q_0\to\alpha(D)/(\Pi R)}\big[D(q_0)-S(q_0)\big] = +\infty$, and that $\lim_{q_0\to\alpha(U)/(1-\chi)}\big[D(q_0)-S(q_0)\big] = -\infty$. Thus, the solution, q_0, to (14) exists and is unique. Given q_0, (9) and (10) determine $\bar{h}_0$ and $\underline{h}_0$ uniquely. Given q_0, $\bar{h}_0$, and $\underline{h}_0$, the portfolio for each type of individuals is determined uniquely as discussed above. This completes the proof. □

2.2 Arrow-Debreu Market Structure

It will prove useful in what follows to compare the price of government bonds, q_0, in the leverage equilibrium to its price when markets are frictionless and agents can trade a complete set of Arrow securities. Since cash is dominated by safe storage, it will not be held in an AD equilibrium and is thus omitted.

Let $q_0(s_1)$ be the price of the Arrow security that pays off one unit of account if and only if state $s_1 \in S$ is realized in period 1. Then the flow budget constraints for agent $h \in [0,1]$ are expressed as

$$\begin{aligned} c_0^h + \frac{q_0}{P_0} b_0^h + k_0^h + \sum_{s_1 \in S} \frac{q_0(s_1)}{P_0} b_0^h(s_1) &\leq e(h) \left(\frac{q_0}{P_0} \bar{B} + y_0 \right), \\ c^h(s_1) \leq \frac{\alpha(s_1)}{P_1} b_0^h + R k_0^h + \frac{1}{P_1} b_0^h(s_1) &+ e(h) \big[y_1 - T(s_1) \big], \qquad s \in S. \end{aligned}$$

Here, $b_0^h(s_1)$, $s_1 \in S$, are the amounts of the Arrow securities purchased by agent h in period 0. All of the other variables are as defined above.

The absence of arbitrage opportunities implies

$$\frac{1}{R} = \sum_{s_1 \in S} q_0(s_1) \frac{P_1}{P_0}, \tag{18}$$

$$\frac{q_0}{P_0} = \sum_{s_1 \in S} \frac{q_0(s_1)}{P_0} \alpha(s_1). \tag{19}$$

Then the lifetime budget constraint for agent h can be written as

$$c_0^h + \sum_{s_1 \in S} \frac{q_0(s_1)}{P_0} P_1 c^h(s_1) \leq \left(\frac{q_0}{P_0} \bar{B} + y_0 \right) + \sum_{s_1 \in S} \frac{q_0(s_1)}{P_0} \big[y_1 - T(s_1) \big] e(h).$$

As in the leverage specification, no agents will consume in period 0: $c_0^h = 0$, for all $h \in [0,1]$. And there is a marginal agent $\bar{h}_0 \in (0,1)$ defined by

$$\frac{1}{q_0(U)} \bar{h}_0 = \frac{1}{q_0(D)} (1 - \bar{h}_0), \tag{20}$$

such that (i) agents with $h \geq \bar{h}_0$ choose to consume only in state U: $c^h(D) = 0$, and

$$\frac{q_0(U)}{P_0} P_1 c^h(U) = \left(\frac{q_0}{P_0} \bar{B} + y_0 \right) + \sum_{s_1 \in S} \frac{q_0(s_1)}{P_0} \big[y_1 - T(s_1) \big] e(h), \tag{21}$$

and (ii) agents with $h < \bar{h}_0$ choose the opposite: $c^h(U) = 0$, and

$$\frac{q_0(D)}{P_0} P_1 c^h(D) = \left(\frac{q_0}{P_0} \bar{B} + y_0 \right) + \sum_{s_1 \in S} \frac{q_0(s_1)}{P_0} \big[y_1 - T(s_1) \big] e(h). \tag{22}$$

The market clearing conditions for government bonds and Arrow securities are:

$$\int_0^1 b_0^h f(h)\, dh = \bar{B},$$
$$\int_0^1 b_0^h(s_1) f(h)\, dh = 0, \qquad s_1 \in S.$$

When combined with the condition that $c_0^h = 0$ for all h, these conditions imply

$$\int_0^1 k_0^h f(h)\, dh = y_0,$$
$$\int_0^1 c^h(s_1) f(h)\, dh = R y_0 + y_1, \tag{23}$$

where the second equality has used the government budget constraint (1). Combining equations (21), (22) and (23), we obtain

$$\left(\frac{q_0(U)}{P_0} P_1\right)^{-1} \left\{ \left(\frac{q_0}{P_0}\bar{B} + y_0\right) \left[1 - G(\bar{h}_0)\right] + \sum_{s_1 \in S} \frac{q_0(s_1)}{P_0} P_1 \left[y_1 - \frac{\alpha(s_1)}{P_1}\bar{B}\right] \left[1 - G(\bar{h}_0)\right] \right\}$$
$$= \left(\frac{q_0(D)}{P_0} P_1\right)^{-1} \left\{ \left(\frac{q_0}{P_0}\bar{B} + y_0\right) G(\bar{h}_0) + \sum_{s_1 \in S} \frac{q_0(s_1)}{P_0} P_1 \left[y_1 - \frac{\alpha(s_1)}{P_1}\bar{B}\right] G(\bar{h}_0) \right\},$$

which can be simplified as

$$\frac{1 - \bar{h}_0}{\bar{h}_0} = \frac{G(\bar{h}_0)}{1 - G(\bar{h}_0)}. \tag{24}$$

The equilibrium for the AD specification is determined as a collection of four variables, $\{q_0, q_0(U), q_0(D), \bar{h}_0\}$, that solves the system of equations given by (18), (19), (20), and (24).

2.3 Analysis of the one-period model

A large body of previous work starting from Miller (1977) has found that imposing short-selling constraints raises asset prices. We will see that in our model increasing the costs of short-selling attenuates the negative response of the government bond price to bad news in period 0. However, it does not immediately follow that the government bond price in our model undershoots the AD price when bad news. A high stock of government debt, a low recovery rate and an initial distribution of $G(\cdot)$ that assigns lots of mass to low h types all have offsetting effects. It is consequently an open question whether the leverage equilibrium bond price on net overshoots or undershoots the AD bond price when bad news arrives about the prospects of a sovereign default crisis. This section contains analytical results. Then in Section we report numerical results for a version of the model that is calibrated to the situation of Greece 2011.

It will prove convenient to define some notation for the lemmas that follow. For given values of the exogenous variables $(\bar{B}, \alpha(D), \chi, \Pi)$ and the distribution function G, the equilibrium values of q_0, $\bar{h}_0$, and $\underline{h}_0$ in the leverage specification can be expressed as $q_0(\bar{B}, \alpha(D), \chi, \Pi|G)$, $\bar{h}_0(\bar{B}, \alpha(D), \chi, \Pi|G)$, and $\underline{h}_0(\bar{B}, \alpha(D), \chi, \Pi|G)$. In the following comparative statics exercises, the partial derivatives of q_0, $\bar{h}_0$, and $\underline{h}_0$ are defined using these functions.[4] Similar notation is used for the Arrow-Debreu specification.

Equivalence of bond prices in the leverage and AD Equilibria We start by considering the leverage equilibrium with no government debt. Even though there is no default in period 1 (taxes are zero) we can still price government debt and agents depending on their beliefs about the event U will choose to take long or short positions. This situation is of interest because the response of bond prices in the leverage equilibrium is the same as in the Arrow-Debreu equilibrium when $\Pi R = 1$ (money earns the safe interest rate and thus the collateral posted by short sellers earns R) and $\chi = 0$.

Lemma 2. *Suppose that $\bar{B} = 0$, $\chi = 0$ and $\Pi R = 1$, then the marginal buyer and the price of government bonds are identical between the leverage and AD specifications*

$$\bar{h}_0 = \underline{h}_0 = \bar{h}_0^{AD}, \quad \textit{and} \quad q_0 = q_0^{AD}. \tag{25}$$

where the superscript AD denotes the equilibrium values in the AD specification.

Proof. Suppose that the hypotheses in the lemma are satisfied. Then, by Lemma 4, we know that $\bar{h}_0 = \underline{h}_0$, and

$$\bar{h}_0 = \frac{q_0 - \alpha(D)}{\alpha(U) - \alpha(D)}. \tag{26}$$

The market clearing condition (14) reduces to

$$q_0 = G(\bar{h}_0)\alpha(D) + \big[1 - G(\bar{h}_0)\big]\alpha(U). \tag{27}$$

Equations (26)-(27) imply that $\bar{h}_0$ is the solution to

$$\bar{h}_0 = 1 - G(\bar{h}_0). \tag{28}$$

Clearly, such a solution exists and is unique. Given $\bar{h}_0$, the government bond price q_0 is determined as:

$$q_0 = \bar{h}_0\alpha(U) + (1 - \bar{h}_0)\alpha(D).$$

[4] There is a slight abuse of notation here. We have defined above the functions $\bar{h}_0(q_0)$ and $\underline{h}_0(q_0)$ using (9) and (10). Our new functions $\bar{h}_0(\bar{B}, \chi, \Pi|G)$ and $\underline{h}_0(\bar{B}, \chi, \Pi|G)$ should be understood as $\bar{h}_0(q_0(\bar{B}, \chi, \Pi|G))$ and $\underline{h}_0(q_0(\bar{B}, \chi, \Pi|G))$.

Now consider the AD equilibrium. The market clearing condition (24) reduces to

$$\frac{1}{q_0^{\mathrm{AD}}(U)}\left[1 - G(\bar{h}_0^{\mathrm{AD}})\right] = \frac{1}{q_0^{\mathrm{AD}}(D)} G(\bar{h}_0^{\mathrm{AD}}). \tag{29}$$

Combining this equation with equation (20) yields

$$\bar{h}_0^{\mathrm{AD}} = 1 - G(\bar{h}_0^{\mathrm{AD}}),$$

which eatablishes that

$$\bar{h}_0^{\mathrm{AD}} = \bar{h}_0.$$

Given that $\Pi R = 1$, it follows from (18)-(19) that

$$\begin{aligned} q_0^{\mathrm{AD}}(U) &= \bar{h}_0^{\mathrm{AD}}, \\ q_0^{\mathrm{AD}}(D) &= 1 - \bar{h}_0^{\mathrm{AD}}, \\ q_0^{\mathrm{AD}} &= \bar{h}_0^{\mathrm{AD}} \alpha(U) + (1 - \bar{h}_0^{\mathrm{AD}}) \alpha(D). \end{aligned}$$

Thus $q_0 = q_0^{\mathrm{AD}}$. This completes the proof. $\square$

Note that even though the prices of government bonds are the same, the allocations are different in the leverage and the AD equilibrium. In the leverage equilibrium agents cannot borrow against their future endowment but in the AD equilibrium they can. If this restriction is relaxed the allocations are also identical in the two market structures.[5]

2.3.1 Variations in parameters that are common to the two market structures

Government debt. Our equivalence result about the price of government debt in the leverage and AD models breaks down when government debt is positive. On the one hand, in the Arrow-Debreu market structure the Ricardian equivalence proposition obtains. Increasing the initial government-debt-output ratio from zero has no effect on q_0. This result follows from the observation that government debt is a redundant security. One the other hand, in the leverage equilibrium both the demand and supply schedules for government debt shift. The supply schedule shifts due to the increase in the supply of government debt. This shift is amplified because short-sellers have access to leverage. The demand schedule also shifts because those who wish to buy bonds also have access to leverage. This second effect is weaker though and the bond price falls when the supply of government debt is increased.

[5]An implication of this final result is that the securities traded in the leverage market structure can span the same space as one period Arrow claims.

Lemma 3. *Given $\bar{B} \geq 0$, $\chi \geq 0$, $\alpha(D) < 1$, and $\Pi \geq R^{-1}$, we have*

$$\frac{\partial q_0}{\partial \bar{B}} < 0, \quad \frac{\partial \bar{h}_0}{\partial \bar{B}} < 0, \quad \frac{\partial \underline{h}_0}{\partial \bar{B}} < 0.$$

Proof. Appendix. □

An immediate implication of Lemma 2 is that there is overshooting in the leverage equilibrium when government debt is in positive net supply $q_0 < q_0^{AD}$.

The pattern of trade in the leverage model depends on the costs of short-selling. Short-selling is costly when either $\chi > 0$ or $\Pi R < 1$. The pattern of trade is simplest when short-selling is costless: $\chi = 0$ or $\Pi R = 1$. From Lemma 3 we know that an increase in government debt, lowers the value of q_0 when $\chi = 0$ and $\Pi R = 1$. However, when both financial frictions are absent all agents participate in the bond market and the identities of the marginal purchaser and marginal short-seller are identical

Lemma 4. *Suppose that $\chi = 0$ and $\Pi R = 1$. Then $\bar{h}_0 = \underline{h}_0$.*

Proof. With $\chi = 0$ and $\Pi R = 1$, equations (9) and (10) become

$$\bar{h}_0 = \frac{q_0 - \alpha(D)}{\alpha(U) - \alpha(D)},$$
$$1 - \underline{h}_0 = \frac{\alpha(U) - q_0}{\alpha(U) - \alpha(D)},$$

which in turn imply that $\bar{h}_0 = \underline{h}_0$. □

It then follows immediately from the observation that q_0 is falling in $\bar{B}$ that the identify of the marginal purchaser of government debt *falls* when $\bar{B}$ is increased when short-selling is costless.

If short-selling is costly, three distinct types of portfolios are held in equilibrium. Some mildly pessimistic individuals prefer to lend at the safe rate of R as compared to taking short positions on government debt. Once the transactions costs of taking short-positions are factored in they receive a higher expected return from lending as compared to short-selling government debt.

A higher level of government debt is associated with more leverage for purchasers of government debt but less leverage for those taking short positions. To see why this obtains recall that leverage for those taking long positions is given by:

$$Lev_{long} = \frac{P_1}{\Pi q_0 - \alpha(D)/R}$$

and that leverage for those taking short-positions is:

$$Lev_{short} = \frac{P_0}{\alpha(U) - (1-\chi)q_0}.$$

Then observe that increasing $\bar{B}$ reduces q_0. Intuitively, q_0 is the cost for those taking long positions and a lower value of q_0 lowers this cost. For short-sellers q_0 is the benefit and this benefit is smaller when q_0 is low.

Given these observations it is straightforward to derive the effects of an increase in government debt on the identity of the marginal purchaser and short seller of government debt when short-selling is costly. Inspection of equations (9) and (10) reveals that an increase in government debt reduces the identity of the marginal purchaser of government debt, $\bar{h}_0$ and also reduces, the identify of the marginal short seller of government debt, $\underline{h}_0$.

Lower recovery rates. Buying debt is a less attractive proposition when the recovery rate is low and it follows that a lower recovery rate lowers the bond price in both the leverage and the AD market structures, as shown in the following lemma.

Lemma 5. *Given $\bar{B} \geq 0$, $\chi \geq 0$, $\alpha(D) < 1$, and $\Pi \geq R^{-1}$, we have*

$$\frac{\partial q_0}{\partial \alpha(D)} > 0, \quad \frac{\partial q_0^{AD}}{\partial \alpha(D)} > 0.$$

Proof. Appendix □

A leftward shift in the distribution of the aggregate endowment by type. Shifting the mass of G towards more pessimistic agents acts to lower the bond price in both the leverage and AD specifications.

Lemma 6. *Given $\bar{B} \geq 0$, $\chi \geq 0$, $\alpha(D) < 1$, and $\Pi \geq R^{-1}$, consider two distribution functions $G^1(h)$ and $G^2(h)$ such that G^2 first-order stochastically dominates G^1. Then the associated leverage equilibrium satisfies*

$$q_0(\cdot|G^1) \leq q_0(\cdot|G^2), \quad \bar{h}_0(\cdot|G^1) \leq \bar{h}_0(\cdot|G^2), \quad \underline{h}_0(\cdot|G^1) \geq \underline{h}_0(\cdot|G^2),$$

and the Arrow-Debreu equilbirium satisfies

$$q_0^{AD}(\cdot|G^1) \leq q_0^{AD}(\cdot|G^2), \quad \bar{h}_0^{AD}(\cdot|G^1) \leq \bar{h}_0^{AD}(\cdot|G^2).$$

Proof. Consider the leverage equilibrium. Corresponding to the distribution function G^i, $i = 1, 2$, let $S^i(q_0)$ and $D^i(q_0)$ be the supply and demand functions for government debt defined in (16)-(17). Since G^2 first-order stochastically dominates G^1, $G^1(h) \geq G^2(h)$ for all $h \in [0,1]$. It follows that for all q_0,

$$S^1(q_0) \geq S^2(q_0), \quad \text{and} \quad D^1(q_0) \leq D^2(q_0).$$

Therefore, $q_0(\cdot|G^1) \leq q_0(\cdot|G^2)$. Then equations (9) and (10) imply that $\bar{h}_0(\cdot|G^1) \leq \bar{h}_0(\cdot|G^2)$ and $\underline{h}_0(\cdot|G^1) \geq \underline{h}_0(\cdot|G^2)$.

In the AD specification, the marginal buyer $\bar{h}_0^{\mathrm{AD}}$ is determined by equation (24), which immediately implies that $\bar{h}_0^{\mathrm{AD}}(\cdot|G^1) \leq \bar{h}_0^{\mathrm{AD}}(\cdot|G^2)$. Equations (18)-(20) imply that

$$q_0^{\mathrm{AD}} = \frac{1}{R\Pi}\left[\bar{h}_0^{\mathrm{AD}} + (1 - \bar{h}_0^{\mathrm{AD}})\alpha(D)\right].$$

It follows that $q_0^{\mathrm{AD}}(\cdot|G^1) \leq q_0^{\mathrm{AD}}(\cdot|G^2)$. This completes the proof. □

We are also not able to establish analytical results about the relative size of the response of q_0 and q_0^{AD}.

2.3.2 Variations in the extent of financial frictions

In the leverage market structure those wishing to take short positions face proportionate transactions costs and are forced to post cash as collateral which is dominated in rate of return when $\Pi R < 1$. We next consider the effects of varying the size of these financial frictions on the price of government debt.

Transactions costs on short-sales of government debt. Miller (1977) shows that costs on short-selling act to raise asset prices when the demand for assets is downward sloping. Our leverage specification has this property. It follows that increasing the costs of short-sales by either increasing χ or lowering π acts to attenuate the response of bond prices to bad news.

Lemma 7. *Given $\bar{B} \geq 0$, $\chi \geq 0$, $\alpha(D) < 1$, and $\Pi \geq R^{-1}$, we have*

$$\frac{\partial q_0}{\partial \chi} > 0, \quad \frac{\partial \bar{h}_0}{\partial \chi} > 0, \quad \frac{\partial \underline{h}_0}{\partial \chi} < 0.$$

Proof. Appendix. □

Intuition for these results can be found by considering changes in the demand and supply schedules for bonds. An increase in χ does not affect the demand for bonds, but shifts the supply curve inward and thus the price of bonds increases.

As the costs of short-selling are increased the identity of the marginal short-seller, $\underline{h}_0$, declines. For short-sellers the increase in bond prices associated with higher transactions costs makes purchasing the bond less attractive proposition and the identity of the marginal purchaser increases.

Those taking long and short positions have the same access to leverage when transactions costs on short-selling are positive. With zero government debt, clearing in the bond market implies that the net supply of government bonds is also zero. This can only occur if short-sellers and long-sellers have the same amount of leverage.

Combining these various responses we see that costly short-sales depresses participation in the market for government bonds. As the costs are increased, the identify of the marginal purchaser increases, the identity of the marginal short-seller falls and an increasing fraction of the population chooses to stay on the side-lines.

Inflation Inflation also makes short-selling costly. To see why this is the case suppose that $\bar{B} = 0$, $\chi = 0$ and $\Pi R > 1$. In this situation cash is dominated in rate of return by safe storage and only short sellers hold cash. They hold cash because it is required as collateral for their short-positions. It follows that inflation acts as a type of transactions tax on short-sellers. A second effect of inflation arises from the fact q_0 is the *nominal* price of bonds. Higher inflation reduces the price of bonds for the standard reasons (it raises the nominal interest rate). This second mechanism is operative in both the Arrow-Debreu and the short-selling market structures. For these reasons it is more meaningful to consider the effect of inflation on the normalized price of bonds, $\rho_0 \equiv q_0 \Pi R$, rather than q_0. Define the function $\rho_0(\bar{B}, \chi, \Pi) \equiv q_0(\bar{B}, \chi, \Pi)\Pi R$.

Lemma 8. *Given $\bar{B} \geq 0$, $\chi \geq 0$, $\alpha(D) < 1$, and $\Pi \geq R^{-1}$, we have*

$$\frac{\partial \rho_0}{\partial \Pi} > 0, \quad \frac{\partial \bar{h}_0}{\partial \Pi} > 0, \quad \frac{\partial \underline{h}_0}{\partial \Pi} < 0.$$

Proof. Appendix. □

The most noteworthy distinction between a higher value of χ and a higher inflation inflation rate is that more inflation is associated with a lower value of q_0 in both market structures. In other respects the results are very simular when either χ or Π is increased.

2.4 Applying the model to Greece

We have identified five factors that influence the magnitude of the decline in bond prices in the leverage and AD market structures. A higher level of government debt results in overshooting of the price of the government bond to bad news in the leverage equilibrium. Transactions costs on short sales and inflation, in contrast, act to attenuate the response of bond prices to bad news. Finally, a leftward shift in the fraction of the aggregate endowment by more pessimistic agents and a reduction in the recovery rate result in larger price responses in both market structures. Which of these effects are most important and on net, how plausible is the possibility that government bond prices are on net under-reacting to bad news about the prospect of a sovereign debt crisis? To get a handle on these questions we consider the combined effects of these five factors on government bond prices in a version of the model that is parameterized to reproduce the situation of Greece prior to the restructuring of its sovereign debt in March of 2012.

Figure 2 reports government bond prices, identities of the marginal agents and leverage at alternative levels of government debt for the leverage and Arrow-Debreu specifications using a parameterization that is based on Greece. The recovery rate is set to 0.5. Our choice is also close to the 53.5% reduction in outstanding principal in Greece's 2012 debt-restructuring agreement. It also falls in the middle of the range of recovery rates on sovereign debt estimated by Moody's prior to Greece's credit event.[6]

The inflation rate is set to 2% which is about the level of the annualized CPI inflation rate of 2.2% that Greece experienced in December 2011. The real interest rate on safe storage is 2%, the proportionate costs of short-selling χ are 3% and the type-endowment distribution G is assumed to be uniform.

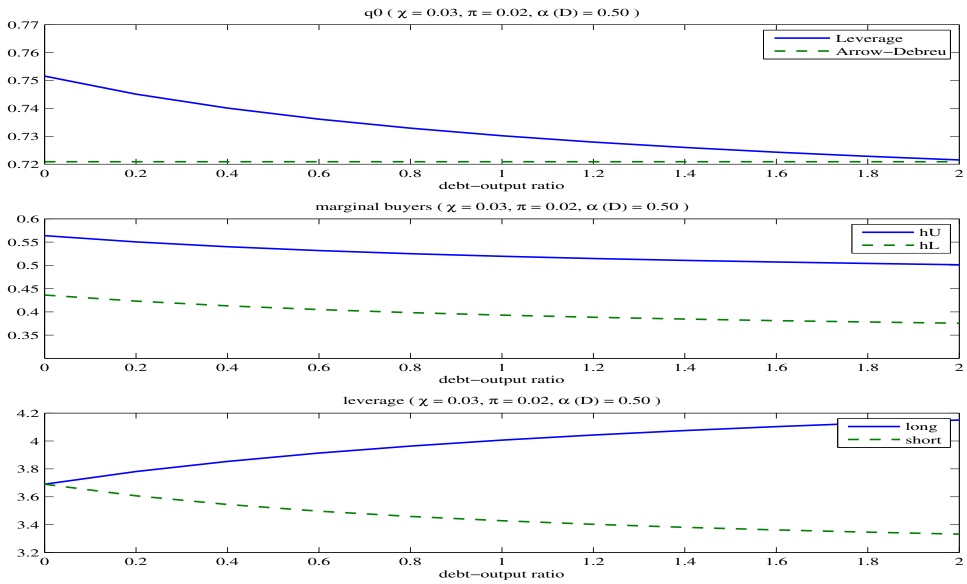
Greece's debt-GDP ratio was 150% in 2010 and 178% in 2011. The results reported in Figure 2 show that for debt-GDP ratios in this range government bond prices in the leverage equilibrium undershoot the Arrow-Debreu bond prices. Indeed, using this parameterization of the model one would conclude that overshooting is a rather remote prospect since it only occurs in the extreme situation where the debt-GDP ratio exceeds two.

As the debt-GDP ratio is increased, the identities of the marginal purchaser and short-seller of government debt both fall for the reasons discussed in Section 2.3.1. The figure reveals that the identities of both marginal agents decline in a proportionate fashion so that the total measure of inactive agents is virtually unchanged at about 12% at both small and large levels of the debt-GDP ratio.

The bottom panel of the figure reports leverage available to each type of agent. As described above more government debt means that there is more of the asset that the optimists wish to take long positions in and their leverage goes up. For short-sellers the converse is the case. The magnitudes range from about 3.3 to 4.2 and may appear to be a bit large. For purposes of comparison leveraged bull and bear U.S. Treasury ETFs typically offer leverage

[6] Moody's estimates range from a low of 0.31 to a high of 0.68, depending on the method, for the years 2001-2003 (see Moody's Global Credit Research March 2008: Sovereign Default and Recovery Rates, 1983-2007).

Figure 2: Government bond prices, marginal purchasers and short-sellers of government debt and leverage at alternative levels of government debt.



multipliers of 2 or 3.[7] This gap though can be attributed to the fact that our definition of leverage is different from the definition used to determine the leverage multiplier in bull/bear exchange traded treasury funds. For instance, bear ETFs offer a daily return that is two or three times the inverse of the daily return of a long-term government bond index. This notion of leverage is given by:

$$Lev_{short}^{ETF} = \frac{q_0}{q_0 - \alpha(D)} \left\{ \frac{\alpha(U) - \alpha(D)}{\Pi\alpha(U) - (1-\chi)\Pi q_0} - 1 \right\}. \tag{30}$$

Under this definition leverage of short-sellers in the model is 2.1 when ($\bar{B}/y_0 = 1.5$, $\alpha(D) = 0.5$, $R = 1.02$, $\pi = 0.02$, $\chi = 0.03$). The Proshares UltraShort 3-7 Year Treasury Bear 2x ETF, offers the same amount of leverage to those interested in taking short positions on U.S. medium term treasuries.[8] This evidence suggests that our choice of $\chi = 0.03$ in conjunction with the other parameters is providing short-sellers with about the right amount of leverage.

Although it is difficult to directly measure the overall costs of short-selling Greek Sovereign

[7]We know of no ETFS that take offer leveraged returns on Greek sovereign debt.

[8]These products are specifically designed for traders with very short planning horizons. The index on these ETF's resets to 100 every day. This renders these products unsuitable for those wishing to use buy and hold strategies to take long-horizon short positions on U.S. government debt.

debt, it is much clearer that they increased as Greece moved towards default. Credit default swaps were traded in Greece well prior its sovereign debt crisis and they did payout when Greek reached an agreement with the EU and IMF to reschedule its debt payments in March 2012. However the net size of the CDS positions was small. At the time of default the net notional amount of CDS contracts was estimated to be about $3 billion. This constituted less than one percent of total outstanding Greek government debt which was about 360 billion euros. One reason the size of CDS positions may have been so small was uncertainty about whether the CDS contracts would pay out at all. In the weeks leading up to the workout efforts were made to structure it in a way so that the workout would not trigger a payout of credit default swaps. Another reason why the volume of CDS positions was low is that European governments had taken previous measures to make short-selling more costly. Germany imposed a ban on naked short sales of foreign sovereign CDSs and financial stocks on May 18, 2010. As conditions worsened in August of 2011, stock prices of Greek banks and other companies plummeted. Greece responded by banning short-sales on all stocks on Aug. 8 2011.[9] France, Italy, Spain and Belgium followed suit shortly thereafter banning short-sales in financial service sector stocks on August 11, 2011. Then in November 2011 the European parliament voted to ban naked CDS on sovereigns.[10]

Figure 3 Illustrates the effects of varying χ in our one-period model. Increasing χ by 2 percentage points attenuates the price response by about 0.08.% This extends the interval of debt-GDP ratios in which the leverage specification produces undershooting from about 2 to 3. Reducing χ from 0.03 to 0.01 has a somewhat smaller effect. The crossing point of the bond price in the leverage and AD equilibria falls from about 2 to about 1.25.

We saw above that a lower value of the recovery ratio in the default state $\alpha(D)$ results in larger price responses in both market structures. But, that analysis did not provide information on the relative magnitudes of the responses. Figure 4 reveals that a marginal reduction in $\alpha(D)$ has a bigger effect on the bond price in the leverage equilibrium as compared to the AD equilibrium when $\alpha(D) < 0.45$. At recovery rates about 0.47, the opposite occurs. Thus, undershooting is more likely to occur at when recovery rates are higher.

Up to this point we have assumed that the distribution of endowments by type is uniform. Figure 4 reports our baseline results using a uniform distribution and two alternative scenarios for $G(h)$. The right-skewed results assume $G(h) = h^2$. This distribution assigns 36% of the total endowment to the most optimistic quintile of h and 4% to the most pessimistic

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[9] Greek banks hold large amounts of Greek government debt and one way to take a short position on a sovereign default is to short Greek banks.

[10] This legislation, however, only came into force in November 2012 well after the Greek CDS credit event.

Figure 3: Price of government debt, (q_0), in the one-period model: Greece Scenario (inflation rate is 2% and the explicit cost of short-selling is set to 1%, 3% or 5%).

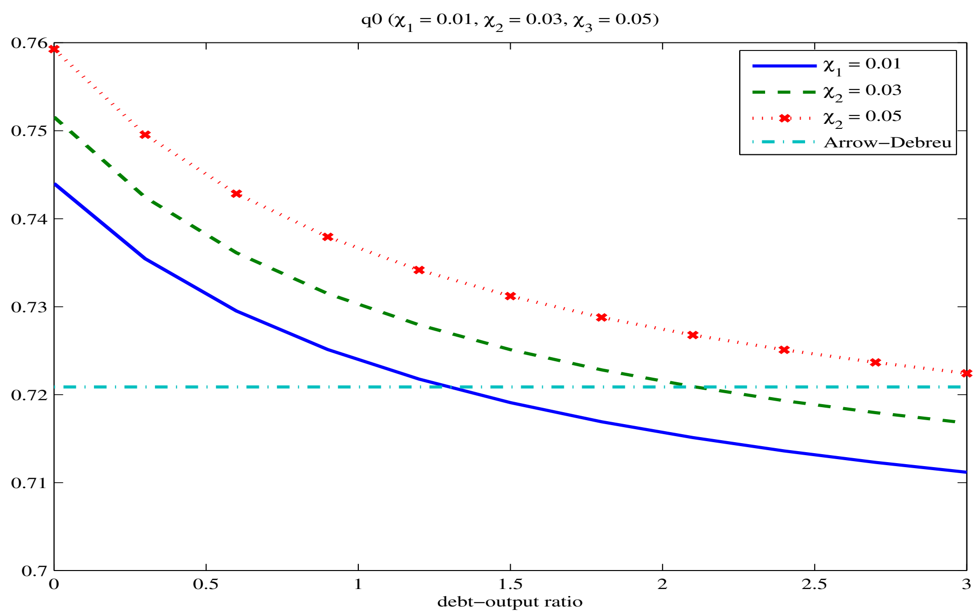


Figure 4: Price of government debt, (q_0), in the one-period model for alternative settings of the recovery rate in the default state $\alpha(D)$.

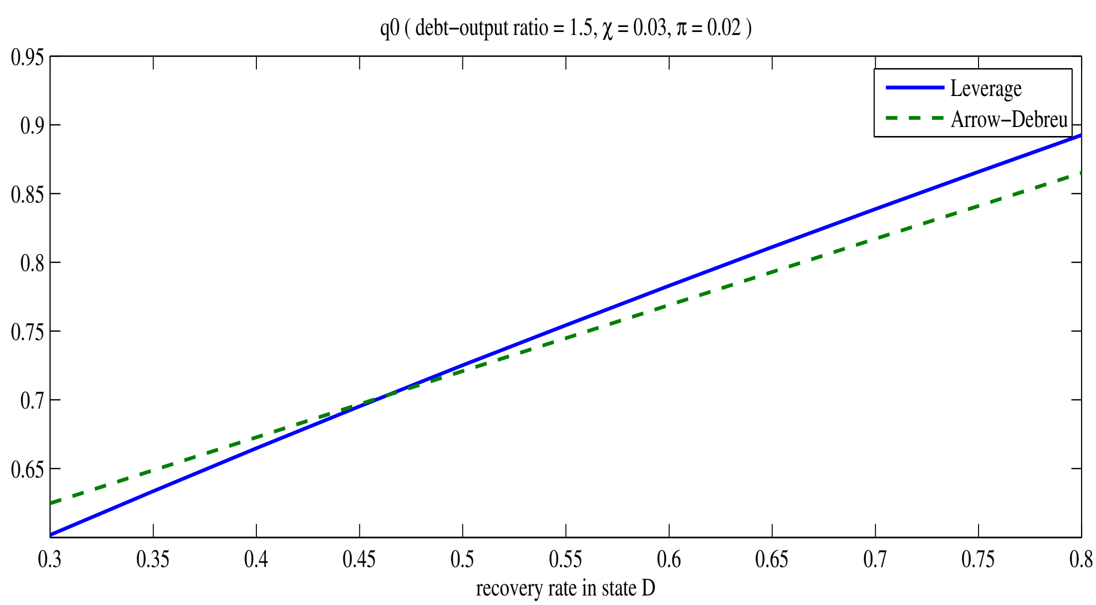
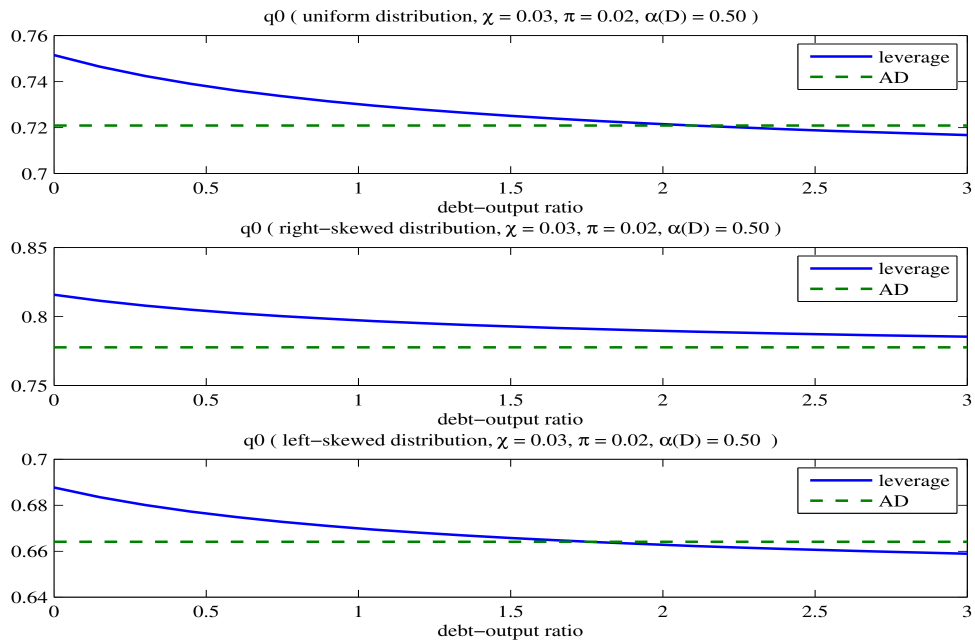


Figure 5: Price of government debt, (q_0), in the one-period model under alternative distributions of endowments by type ($G(h)$): Greece Scenario (inflation rate is 2% and explicit cost of short-selling is 3%).



quintile. Shifting the mass of G towards the optimists increases the bond price in both market structures. However, the price increase is higher in the leverage market structure which means that undershooting occurs for a larger range of values of the debt-GDP ratio.

The left skewed results reported in the lower panel assume that $G(h) = 2h - h^2$. This distribution assigns 4% of the mass to the most optimistic quintile and 36% to the most pessimistic quintile. With a more pessimistic population, the bond price exhibits larger declines in response to the bad news under both market structures and overshooting now occurs in the leverage equilibrium when $\bar{B}/y0$ exceeds 1.7.

Taken together these results indicate that there are large regions of the parameter space where the response of the government bond price is to bad news is attenuated by costs on short-selling. These effects are most pronounced when, $\alpha(D) > 0.47$, less optimist agents receive a larger share of the aggregate endowment and the costs of taking short-selling are 3% or higher.

Table 1: Percentage of the population that goes bankrupt in each history in the one-period model

Scenario		$\chi = 0.0$	$\chi = 0.05$	$\chi = 0.13$	No short sales
1. $\alpha(D) = 0.5$, $\pi = 0.02$					
	U	41	36	29	0
	D	51	48	43	24
2. $\alpha(D) = 0.5$, $\pi = -0.02$					
	U	46	41	34	0
	D	54	51	46	24

*All numbers are percentages of the total population.

2.5 Welfare

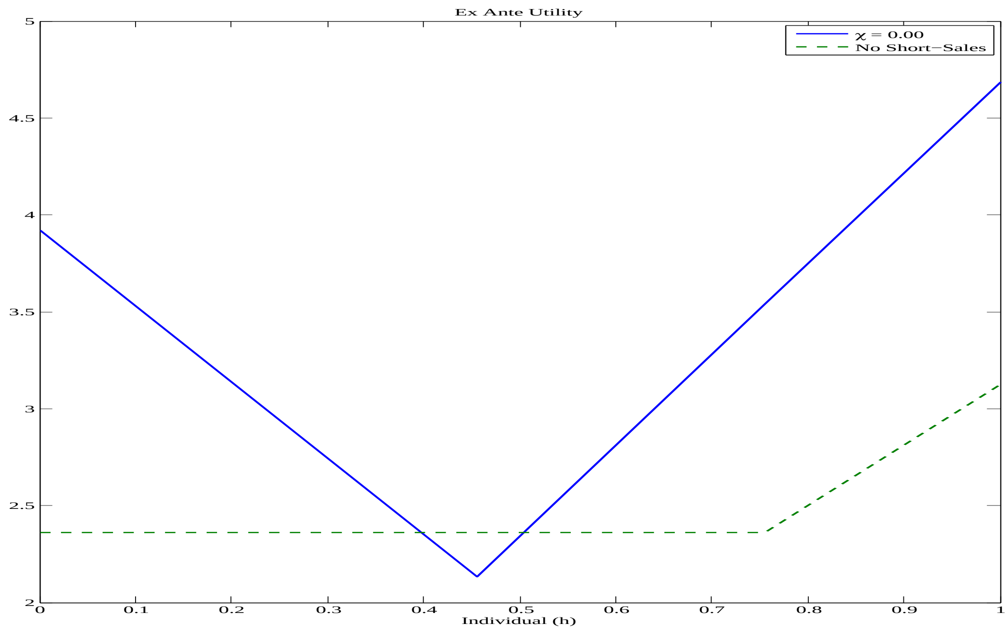
Conducting welfare comparisons in our model is subtle because agents have heterogenous beliefs and it is not clear how one should aggregate these beliefs. We start by considering a Rawlsian notion of welfare and document the ex post percentage of the population that goes bankrupt. Up to now we have only considered the single history that results in default. Table 3 reports the percentage of the population that is bankrupt in period 2 for alternative parameterizations of the 1- period model. Recall that a realization of D results in bankruptcy of all agents who took long positions. This follows from the fact that agents are risk neutral. A realization of U results in bankruptcy for all agents who took short positions. It is also useful to keep in mind that a realization of U results in higher taxes in period 1.

The results reported in Table 1 indicate that there is a strong rationale for imposing short-selling constraints under the Rawlsian welfare criterion. The percentage of the bankrupt population falls monotonically as χ is increased from zero. This pattern occurs in both the states where default occurs and in the state where there is no default. As the costs of short-selling rise, a larger percentage of the population stays on the side-lines. Instead of participating in the bond market, they choose to make safe loans to those taking leveraged-long positions instead.

Brunnermeier, Simsek and Xiong (2014) propose an alternative Bergsonian welfare criterion for aggregating individual utilities when agents have heterogenous beliefs.[11] In our model

[11]See Definition 1 in Brunnermeier et al. (2014).

Figure 6: Ex ante welfare by type of individual (h) in the one-period model with $\pi = -0.02$ and a debt-output level of 1.5.



increasing χ from zero results in a loss in resources due to the fact that every short-seller faces a loss in goods whenever short-selling occurs in equilibrium. When inflation exceeds the Friedman rate, short-sellers also face a cost when $\chi = 0$ because they have to hold their collateral in the form of cash. Only a complete ban on short-selling avoids this cost and it follows that both the Brunnermeier et al. (2014) welfare criterion and the Rawlsian criterion select the no-short-selling specification.

If the inflation rate is set according to the Friedman rule instead, both the $\chi = 0$ and the no short-selling specifications are selected by the Brunnermeier et al. (2014) welfare criterion. Utility is linear in our model and aggregate payouts are identical both when transactions costs on short-selling are infinite and when there are no costs of short-selling. It is interesting that their welfare criterion gives the same ranking to an allocation in which the minimum percentage of the population that goes bankrupt is 46% (no costs of short-selling, $\pi = -0.02$) and to an allocation where the maximum percentage of the population that goes bankrupt is 24% (a total ban on short-selling, $\pi = -0.02$).

We conclude this section by documenting properties of ex ante welfare for each individual in our economy. We limit attention to the no-short-selling scenario and the scenario with ($\chi = 0$ and $\Pi = 1/R$) and a debt-GDP ratio of 1.5. Agents with h sufficiently low always prefer the specification with no costs on short-sales. We have explained above that costs on short-sales also increase costs for those wishing to take long positions and it follows that those with h sufficiently high will also prefer the economy with costless short-selling. However, for h close to 1/2, these benefits are smaller and they in fact prefer the economy with no short-sales in some situations. For instance, in the one-period model agents with $0.4 \leq h \leq 0.5$ have higher expected utility in the economy with no short-sales (see Figure 6).

3 The T-Period Model

We now turn to consider the T-period model with explicit default. Extending the number of periods allows us to analyze the dynamics of government bond yield movements leading up to a sovereign default. We assume that a model period corresponds to a year and when we calibrate the model we calibrate it using annualized interest rates. Generalizing the model in this way makes it possible to analyze the dynamics of bond price movements leading up to a sovereign debt crisis and in particular to illustrate situations where the initial response of bond prices to bad news about sovereign default is very small.

3.1 The Model

Let $\bar{B}$ be the face value of government debt in period 0 and suppose that the government does not issue new debt in any other period. All government debt is long-term and matures in the final period, T. Suppose also that the government only collects taxes in the last period. Under these assumptions the nominal outstanding value of government debt is $\bar{B}$ in all but the last period.

As before, a shock $s_t \in \{U, D\}$ is realized in periods $t = 1, \ldots, T$. The government defaults in period T only if $s^T = D^T$; it repays the full amount of $\bar{B}$ otherwise. When the government defaults, it repays only a fraction $\alpha \in (0, 1)$ of $\bar{B}$. It follows that the amount of taxes collected in the last period, $T(s^T)$, is given by

$$T(s^T) = \alpha(s^T) \frac{\bar{B}}{P(s^T)},$$

where $\alpha(s^T)$ is defined as

$$\alpha(s^T) = \begin{cases} \alpha, & \text{if } s^T = D^T, \\ 1, & \text{otherwise.} \end{cases}$$

It is convenient to write $q(s^T) = \alpha(s^T)$.

The flow budget constraints for agent h are given by

$$
\begin{aligned}
&c_0^h + k_0^h + \frac{q_0}{P_0} b_0^h + \frac{m_0^h}{P_0} \le \frac{q_0}{P_0} \bar{B} + y_0 + \phi_0^h - \chi_0^h, \\
&c^h(s^t) + k^h(s^t) + \frac{q(s^t)}{P_t} b^h(s^t) + \frac{1}{P_t} m^h(s^t) \\
&\qquad \le \frac{q(s^t)}{P_t} b^h(s^{t-1}) + R[k^h(s^{t-1}) - \phi^h(s^{t-1})] + \frac{1}{P_t} m^h(s^{t-1}) + \phi^h(s^t) - \chi^h(s^t), \quad t = 1, \dots, T-1,\, s^t \in S^t, \\
&c^h(s^T) = \frac{q(s^T)}{P_T} b^h(s^{T-1}) + R[k^h(s^{T-1}) - \phi^h(s^{T-1})] + \frac{1}{P_T} m^h(s^{T-1}) + y_T - T(s^T), \qquad s^T \in S^T,
\end{aligned}
$$

where the short-selling fees are

$$
\chi^h(s^t) = \chi \max\left(0, -\frac{q(s^t)}{P_t} b^h(s^t)\right).
$$

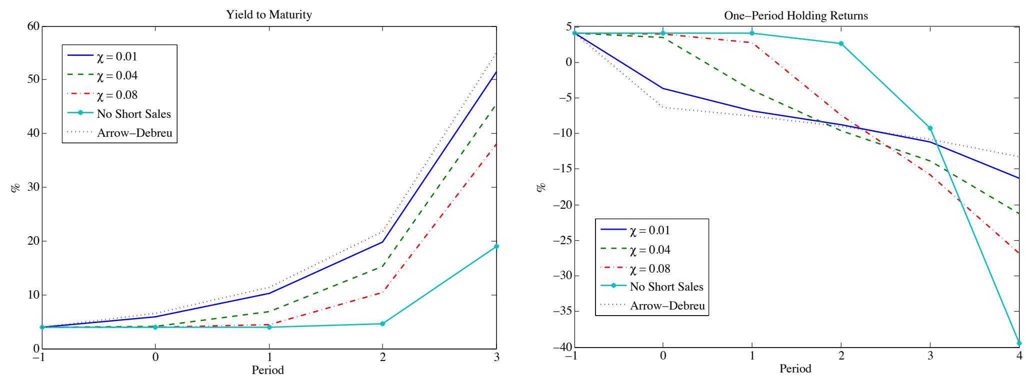
This structure has the property that the short-selling fees are paid every period. We believe that this is a reasonable assumption. ETFs that are used to short-sell U.S. treasuries, for instance, reset to one hundred either every day or every month. An investor who uses this security to take a short-position at a longer horizon must readjust his portfolio on a daily or monthly basis which incurs transactions costs. Similarly, open positions in options and futures contracts are concentrated at very short-horizons of less than six months. Rolling over these contracts to take short-positions are longer horizons also incurs transactions costs. The collateral constraints are:

$$
\begin{aligned}
&R\phi^h(s^t) \le \frac{q(s^{t+1})}{P_{t+1}} b(s^t), \qquad t = 0, \dots, T-1,\, s^{t+1} \in S^{t+1},\, s_{t+1} \in S, \\
&m^h(s^t) \ge -q(s^{t+1}) b^h(s^t)
\end{aligned}
$$

The non-negativity constraints: $c(s^t), k(s^t), m(s^t) \ge 0$ for all t and s^t.

The T-period model is sufficiently complex that we have no alternative but to rely entirely on computational methods in characterizing the equilibrium. Finding the marginal purchasers and sellers of government debt is a rather subtle numerical problem that quickly becomes intractable as the number of periods increases and the number of potential trading strategies for each individual increases. Using numerical methods though we have verified that the T-period model has the same basic properties as the one-period model. Increasing the supply of government debt, lowering the recovery rate and shifting the distribution of endowments to the left all act the increase the bond price response to bad news in the leverage model. Higher costs of short-selling and higher inflation have the opposite effect.

Figure 7: Yields and one-period holding returns on government bonds in the 4-period model with explicit default at alternative costs of short-selling, χ. Greece scenario: inflation rate is 2%, debt-GDP ratio is 1.5 and the recovery rate is 0.5.



(a) Yield to maturity of a government bond that matures in period 4.

(b) One period holding returns on government debt.

3.2 The T-period applied to Greece

Our model has rich implications for the dynamics of bond price movements and the pattern of trade. We now illustrate these properties using our parameterization for Greece, ($R = 1.02$, $\pi = 0.02$, $\alpha(D) = 0.5$, $\bar{B}/y = 1.5$, $G \sim U(0, 1)$).

Price Dynamics We start by considering how costs on short-sales affect the dynamics of bond yields and one-period holding returns along the path to default in our model with explicit default. Figure 7a reports yields to maturity of a government bond that matures in the final period in a 4-period version of the model.[12] Results are reported for the Arrow-Debreu market structure, the leverage market structure with no short sales and three intermediate settings of χ. All of the results use our previous Greece scenario with a debt-output ratio of 1.5 and an inflation rate of 2%.

Our first finding is that costly short-selling results in lower bond-yields than the Arrow-Debreu benchmark. This occurs for all values of $\chi > 0$ and in all periods leading up to default.

In the one-period model with explicit default we saw that a higher debt-GDP ratio in isolation acted to produce overshooting and that the size of this effect could be quite significant. The range of model parameters that produce undershooting of government bond yields is much larger in the multi period model. For instance, using a two percent inflation rate and

[12] To be more precise the figure reports $1/(T - t)\ln(1/q(s^t))$ with $T = 4$ and $t = -1, 0, \ldots, 3$.

$\chi = 0.01$, overshooting occurs in the one-period model when the debt-GDP ratio is larger than 1.5 with $\chi = 0.01$. Using the same values of inflation and χ, the debt-output ratio has to be in excess of 30 to produce overshooting when $T = 4$. This finding is due to the fact that adding more periods reduces the overall level of short-selling activity and this limits the scope for overshooting.

Observe next that the extent of overshooting as measured by the vertical distance between the yield in the Arrow-Debreu market structure and the short-selling market structure increases along the path leading to default. When $\chi = 0.01$. the extent of undershooting varies from a low of 71 basis points in period 0 to a high of 346 basis points in the period immediately prior to default. As χ is increased the magnitude of undershooting relative to the Arrow-Debreu benchmark increase. For instance, the gap between the two market structures is 958 basis points in period three when $\chi = 0.04$ and 3596 basis points in period three when short-sales are not allowed.

Note next that as χ is increased the initial response of bond yields drops to zero. For instance, when $\chi = 0.04$ the yield of the government bond rises by only 15 basis points in period zero and when $\chi = 0.08$ there is no response in the bond yield in either period zero or period one. Instead, responses get delayed and concentrated into states that are close to a sovereign default.

The fact that costly short-selling is associated with undershooting in government bond yields in all periods might appear to be at odds with our previous results for the model with implicit default. In that model the inflation rate exhibited undershooting relative to the same Arrow-Debreu benchmark in early periods and overshooting shortly before default. This difference between the two models has to do with the fact that bond yields and inflation are different prices. Inflation in the model with implicit default corresponds to one period holding returns in our model with explicit default. Figure 7b displays one-period holding returns which are given by $q(s^{t+1})/q(s^t)$. From this figure we see that one-period holding returns exhibit the same pattern of under-shooting in early periods and over-shooting in states close to default that we found in the model with implicit default. All of the leverage market structures undershoot the Arrow-Debreu returns in the first two periods and overshoot in the final period.

In our model with implicit default no short-selling occurred in equilibrium and we were thus not able to document how the dynamics of the model change as the costs of short-selling are altered. It is very clear from the results in Figures 7a-7b that even moderate transactions costs of a magnitude from 4-8% significantly alter the price dynamics in a way that makes them resemble the dynamics of the model when no short-sales are banned.

Overall, short-selling costs act to delay and concentrate the response of bond yields to bad news. When $\chi \geq 0.04$ bond yields and one-period holding returns exhibit no discernible response to the initial bad news in period 0. The responses of both variables also exhibit a discernible delay as compared to Arrow-Debreu. Bond yields undershoot the Arrow-Debreu reference point in all periods leading up to default but, one-period holding returns overshoot in states immediately prior to default.[13]

Trading Dynamics The multi-period model with explicit default also has a rich set of implications for the dynamics of trade. Consider Table 2 which reports the percentage of the population taking short and long positions on government debt and the total percentage of the population that is participating in the government bond market in each period for the history that results in a sovereign default. We saw above that transactions costs on short-selling reduced participation of short-sellers in the government bond market in the one-period model. This same phenomenon occurs in the T-period model. When short-selling constraints are absent 50% of the population takes short-positions along the path leading to a sovereign default (see row 5 of Table 2).[14] This percentage drops to 34.3% when $\chi = 0.0$ and is due to the fact that inflation also increases the costs of taking-short positions as we documented in the one-period model above. As χ is increased, the percentage of the population taking short positions falls to 29.6% when $\chi = 0.01$ and to 21.5% when $\chi = 0.04$.

The effect of the transactions cost on short-selling is most pronounced in early periods when the prospect of a default is distant. No agents choose to take short positions in period zero when $\chi = 0.04$, and no agents take short positions in either period 0 or period 1 when $\chi = 0.08$. As default approaches short-selling activity increases. For instance, short-selling is monotonically increasing as default approches when $\chi = 0.08$. This pattern is the mirror opposite of the Arrow-Debreu equilibrium which exhibits a monotonic declines in short-sales activity along the path to default.

Transactions costs on short-sales also have a depressing effect on the participation of purchasers of government debt. In the Arrow-Debreu market structure of 80% of the population takes a long position on government bonds at some point along this path.[15] In the leverage

[13]These results may appear to differ from results in Geanakoplos (2010). However, that is not the case. For instance, if we add 2 more periods to the example reported in Section 3 of his paper the resulting sequence of prices is $q = [0.999996, 9972, 9326, 0.68, 0.2]$. From this we see that the response of the price to bad news in the first two periods is very small, but that the response to bad news in the final period is very large.

[14]The "total" percentage of the population that takes a short position is the maximum percentage of short-sellers in the given column.

[15]The " total" percentage of the population taking long positions is the sum of long-purchasers in each row

Table 2: Traders in government bonds in the 4-period model with explicit default for the history that results in a sovereign default.

Period	History	Arrow-Debreu	$\chi = 0.0$	$\chi = 0.01$	$\chi = 0.04$	$\chi = 0.08$	No short sales
Short in government bonds							
0	0	50.00	34.34	29.58	0.00	0.00	0.00
1	D	33.33	20.31	19.46	21.49	0.00	0.00
2	D^2	25.00	15.98	15.47	19.12	20.13	0.00
3	D^3	20.00	15.43	14.90	18.70	22.80	0.00
Total Short		50.00	34.34	29.58	21.49	20.13	0.00
Long in government bonds							
0	0	50.00	31.52	28.71	4.30	0.71	0.01
1	D	16.67	18.45	18.40	23.10	6.10	0.77
2	D^2	8.33	15.09	15.01	19.89	23.89	7.10
3	D^3	5.00	11.43	13.66	20.65	27.03	20.19
Total Long		80.00	76.49	.75.78	67.94	57.73	28.07
Total active traders							
0	0	100.00	65.86	58.29	4.30	0.71	0.01
1	D	50.00	38.76	37.86	44.59	6.10	0.77
2	D^2	33.33	31.07	30.48	39.01	44.02	7.10
3	D^3	25.00	26.86	28.56	39.35	49.83	20.19

*All numbers are percentages of the total population. Total active traders refers to all agents who participate in the government bond market and is the sum of those taking long and short positions. Total Short (Long) refers to the total percentage of the population that takes a short (long) position along this history.

market structure participation of those taking long positions falls to 76.5% when $\chi = 0$, 67.9% when $\chi = 0.04$ and 28.1% when short-sales are banned entirely.

The reason for this decline in long positions was discussed in the one-period model. As χ increases, the supply of bonds available to those taking long positions falls and leverage available to those wishing to take long positions declines. This in turn increases the identity of the marginal purchaser of government debt.

Using our 4-period model we can document the dynamics of this general equilibrium effect of costly short sales on long bond trading. The depressing effect of costly short-selling long bond trading is most pronounced in early periods. For instance, when $\chi = 0.04$ only 4.3% of the population takes long positions in government debt on period 0 as compared to 50% in the Arrow-Debreu equilibrium. Costly short-sales also disrupt the timing of long-trading. When $\chi \geq 0.08$. the fraction of agents taking long positions increases monotonically as default approaches. Whereas in the Arrow-Debreu equilibrium the fraction falls monotonically.

The final four rows of Table 2 reports the total number of active traders in the government bond market in each period. As the costs of short-selling rise, an increasing fraction of the population chooses to stay on the sidelines and hold their assets in the form of safe storage or equivalently safe loans to traders taking long-positions.

The percentage of agents that are on the sidelines is largest in early periods. Only 4.3% of the population has an active trading position in period zero when $\chi = 0.04$ and only 0.71% is active when $\chi = 0.08$. However, as the economy moves closer to a sovereign default, the returns from betting on this event increase and the percentage of the population taking both short and long positions increases. For the case of $\chi = 0.04$, 39.4% of the population participates in the government bond market in period 3 (state D^3) with 20.7% taking long positions and the remaning 18.7% taking short positions. Higher transactions costs act to delay and concentrate trading activity into states closer to the sovereign-default. However, this effect is not monotonic. Trading activity in period 3 is higher, for instance, when $\chi = 0.08$ as compared to short-sales are ruled out. Note also that when this burst trading activity occurs its scale can be so large as to exceed the overall level of trading activity in the Arrow-Debreu market structure. This can be seen either when $\chi = 0.04$ or 0.08. Higher participation is in turn is associated with the sudden and large movements in bond yields and one-period holding returns in periods two and three that we documented above.

Short-selling activity is often attributed to the large price swings that occur shortly before sovereign debt crises and also exchange rate crisis. When $\chi \geq 0.04$, participation of short-sellers is increasing but participation of those taking long-positions is increasing by even more

of a given column.

Table 3: Percentage of the population that goes bankrupt in each history in the 4-period model with explicit default.

Period	History	$\chi = 0.0$	$\chi = 0.01$	$\chi = 0.04$	$\chi = 0.08$	No short sales
1	U	34.34	29.58	0.00	0.00	0.00
2	D, U	51.83	48.17	25.79	0.71	0.01
3	D, D, U	65.95	62.58	46.52	26.94	0.78
4	D, D, D, U	80.49	77.02	65.99	53.5	7.88
4	D, D, D, D	76.49	75.78	67.94	57.73	28.07

*All numbers are percentages of the total population. A sovereign debt crisis does not occur in any history in which the event U occurs. We thus terminate the history at the point that the U event is realized.

in every period along this path.

Increasing the costs of short-selling We next consider the effects of increasing the costs of short-selling as was done by Europe when it banned naked short-sales of sovereign debt in 2011-2012. Short-sellers face a range of costs even in normal times (see Angel (2004)). Using our short-selling specification. It thus makes sense to recognize this fact and start from a baseline where the transactions costs of short-selling are positive. We have already seen that bond yields are lower when the costs of short-selling are positive. The results in Figure 7a also indicate that increasing the costs of short-selling as bond yields start to rise in anticipation of a possible sovereign debt crisis substantially reduces the magnitude of the increase in bond yields. For instance, imposing restrictions on short-selling that are equivalent to increasing the transactions cost χ from 1% to 4% reduces government bonds yields by 180 basis points in period 0 if the restrictions are imposed at that juncture. The differences are even larger if the same measure is taken in subsequent periods. In our model a sovereign-default is an exogenous event that does not depend on funding costs. Still, these results suggest that the savings to the fiscal authority from this type of measure could be substantial.

Welfare The case for banning short-sales is, if anything even stronger in the 4-period model using the Rawlsian notion of welfare. Table 3 reports the percentage of the population that is bankrupt for each history with a 2% inflation rate. Recall that a realization of D results in bankruptcy of all agents who took long positions at the previous stage. This follows from the fact that agents are risk neutral. A realization of U results in bankruptcy for all agents who took short positions in the previous stage. It is also useful to keep in mind that we have assumed that any history in which a realization of U results in higher taxes in the final period.

A comparison of the results reported in Table 3 with those in Table 1 indicate that there is an even stronger rationale for imposing short-selling constraints. The maximum fraction of the population that experiences losses is even higher in this version of the model. And there continues to be a large gap in welfare across all histories when comparing the specification with a total ban on short-selling to the other specifications.

We have also computed welfare using the ex-ante welfare criterion. Now all agents prefer the economy with $\chi = 0$ to the economy with no-short sales.

4 Conclusion

The world has recently witnessed a number measures taken by governments that have increased the cost of short-selling or even banned short-selling entirely. We have found that these restrictions disrupt a basic price-revelation mechanism associated with forward looking behavior. In frictionless markets bad news about future outcomes gets reflected in prices today as individuals trade on the news. Our findings suggest that the action of short-sellers plays an essential role in this price-revelation mechanism. Small transactions costs on short-sellers of a magnitude ranging from 4 to 8% severely disrupt this mechanism. An outright ban on short-sales of government debt has an even more pronounced effect on bond price and inflation dynamics.

In the context of our model there are two justifications for a government to impose costs on short-selling. First, higher costs on short-selling government debt reduces downward price pressure on government debt in the short-run. This short-run benefit has a cost. When prices do move, the movements are more sudden and large. Second, higher costs of short-selling reduce participation in government bond markets and this in turn reduces the fraction of agents that go bankrupt.

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A Appendix

A.1 Proof of Lemma 3

We first determine the sign of $\frac{\partial q_0}{\partial \bar{B}}$. As in the main text, define the functions $\bar{h}_0(q_0)$ and $\underline{h}_0(q_0)$ using equations (9) and (10):

$$\bar{h}_0(q_0) \equiv \frac{1}{\alpha(U)-\alpha(D)}\big[\Pi R q_0 - \alpha(D)\big], \tag{31}$$

$$\underline{h}_0(q_0) \equiv 1 - \frac{\Pi R}{\alpha(U)-\alpha(D)}\big[\alpha(U) - (1-\chi) q_0\big]. \tag{32}$$

Then write the market clearing condition (14) as

$$\frac{\bar{B}}{P_0} + G_0\big[\underline{h}_0(q_0)\big]\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\alpha(U)-(1-\chi)q_0} = \big(1 - G_0\big[\bar{h}_0(q_0)\big]\big)\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}}.$$

Differentiating this function with respect to $\bar{B}$ and q_0, we obtain

$$\begin{aligned}
&\frac{d\bar{B}}{P_0}\left\{1 + G_0(\underline{h}_0)\frac{q_0}{\alpha(U)-(1-\chi)q_0}\right\} \\
&\quad + dq_0\left\{g(\underline{h}_0)\underline{h}_0'(q_0)\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\alpha(U)-(1-\chi)q_0}\right. \\
&\qquad\qquad \left. + G_0(\underline{h}_0)\left[\frac{\frac{\bar{B}}{P_0}}{\alpha(U)-(1-\chi)q_0} + \frac{(1-\chi)\left(y_0 + q_0\frac{\bar{B}}{P_0}\right)}{\big(\alpha(U)-(1-\chi)q_0\big)^2}\right]\right\} \\
&= \frac{d\bar{B}}{P_0}\big[1 - G_0(\bar{h}_0)\big]\frac{q_0}{q_0 - \frac{\alpha(D)}{\Pi R}} \\
&\quad + dq_0\left\{-g(\bar{h}_0)\bar{h}_0'(q_0)\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}}\right. \\
&\qquad\qquad \left. + \big[1 - G_0(\bar{h}_0)\big]\left[\frac{\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} - \frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\left(q_0 - \frac{\alpha(D)}{\Pi R}\right)^2}\right]\right\},
\end{aligned}$$

which can be rearranged as

$$\begin{aligned}
&\frac{d\bar{B}}{P_0}\left\{1 + G_0(\underline{h}_0)\frac{q_0}{\alpha(U)-(1-\chi)q_0} - \big[1 - G_0(\bar{h}_0)\big]\frac{q_0}{q_0 - \frac{\alpha(D)}{\Pi R}}\right\} \\
&\quad = dq_0\left\{-g(\underline{h}_0)\underline{h}_0'(q_0)\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\alpha(U)-(1-\chi)q_0} - G_0(\underline{h}_0)\frac{\alpha(U)\frac{\bar{B}}{P_0} + (1-\chi)y_0}{\big[\alpha(U)-(1-\chi)q_0\big]^2}\right. \\
&\qquad\qquad \left. - g(\bar{h}_0)\bar{h}_0'(q_0)\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} - \big[1 - G_0(\bar{h}_0)\big]\frac{y_0 + \frac{\alpha(D)}{\Pi R}\frac{\bar{B}}{P_0}}{\left(q_0 - \frac{\alpha(D)}{\Pi R}\right)^2}\right\}
\end{aligned}$$

It is straightforward to see that the coefficient on dq_0 in this equation is negative. We shall show that the coefficient on $d\bar{B}/P_0$, x_1, is positive:

$$x_1 \equiv 1 + G_0(\underline{h}_0)\frac{q_0}{\alpha(U) - (1-\chi)q_0} - \big[1 - G_0(\bar{h}_0)\big]\frac{q_0}{q_0 - \frac{\alpha(D)}{\Pi R}}.$$

For this, rewrite the market clearing condition (14) as

$$x_1\frac{\bar{B}}{P_0} = x_2 y_0,$$

where

$$x_2 \equiv \frac{G_0(\underline{h}_0)}{\alpha(U) - (1-\chi)q_0} - \frac{1 - G_0(\bar{h}_0)}{q_0 - \frac{\alpha(D)}{\Pi R}}.$$

Note that

$$x_1 = 1 - x_2 q_0. \tag{33}$$

Remember that $y_0 > 0$. If $\bar{B} = 0$, x_2 must be zero and thus $x_1 = 1 > 0$. If $\bar{B}/P_0 > 0$, both x_1 and x_2 must be nonzero and have the same sign:

$$\text{sign}(x_1) = \text{sign}(x_2).$$

From (33), this is possible only if $x_1, x_2 > 0$. Therefore, $x_1 > 0$, and thus $\partial q_0/\partial\bar{B} < 0$. It then follows from (31)-(32) that $\partial\bar{h}_0/\partial\bar{B} < 0$ and that $\partial\underline{h}_0/\partial\bar{B} > 0$.

A.2 Proof of Lemma 5

We first show that $\frac{\partial q_0}{\partial\alpha(D)} > 0$. In this exercise, all parameters except for $\alpha(D)$ are held fixed. Thus, using (9)-(10), express $\bar{h}_0$ and $\underline{h}_0$ as functions of q_0 and $\alpha(D)$:

$$\begin{aligned}
\bar{h}_0(q_0, \alpha(D)) &\equiv \frac{1}{\alpha(U) - \alpha(D)}\big[\Pi R q_0 - \alpha(D)\big], \\
\underline{h}_0(q_0, \alpha(D)) &\equiv 1 - \frac{\Pi R}{\alpha(U) - \alpha(D)}\big[\alpha(U) - (1-\chi)q_0\big].
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\partial\bar{h}_0}{\partial q_0} &= \frac{\Pi R}{\alpha(U) - \alpha(D)} > 0, \qquad (\because \alpha(D) < \alpha(U) \equiv 1), \\
\frac{\partial\bar{h}_0}{\partial\alpha(D)} &= \frac{\Pi R q_0 - \alpha(U)}{\big[\alpha(U) - \alpha(D)\big]^2} < 0, \qquad (\because \Pi R q_0 < \alpha(U)), \\
\frac{\partial\underline{h}_0}{\partial q_0} &= \frac{\Pi R(1-\chi)}{\alpha(U) - \alpha(D)} > 0, \qquad (\because 0 \le \chi < 1), \\
\frac{\partial\underline{h}_0}{\partial\alpha(D)} &= -\frac{\Pi R\big[\alpha(U) - (1-\chi)q_0\big]}{\big[\alpha(U) - \alpha(D)\big]^2} < 0, \qquad (\because q_0 < \alpha(U)).
\end{aligned}$$

Then write the market clearing condition (14) as

$$\frac{\bar{B}}{P_0} + G_0\big[\underline{h}_0(q_0, \alpha(D))\big]\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi)q_0} = \big(1 - G_0\big[\bar{h}_0(q_0, \alpha(D))\big]\big)\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}}.$$

Differentiating this function with respect to $\alpha(D)$ and q_0, we obtain

$$\begin{aligned}
& d\alpha(D)\left\{g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial \alpha(D)}\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi)q_0}\right\} \\
&\quad + dq_0\left\{g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial q_0}\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi)q_0} + G(\underline{h}_0)\left[\frac{\frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi)q_0} + \frac{(1-\chi)\left(y_0 + q_0\frac{\bar{B}}{P_0}\right)}{\big[\alpha(U) - (1-\chi)q_0\big]^2}\right]\right\} \\
&= d\alpha(D)\left\{-g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial \alpha(D)}\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} + \big[1 - G(\bar{h}_0)\big]\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\left[q_0 - \frac{\alpha(D)}{\Pi R}\right]^2}\frac{1}{\Pi R}\right\} \\
&\quad + dq_0\left\{-g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial q_0}\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} + \big[1 - G(\bar{h}_0)\big]\left[\frac{\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} - \frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\left[q_0 - \frac{\alpha(D)}{\Pi R}\right]^2}\right]\right\}.
\end{aligned}$$

This can be rearranged as

$$\begin{aligned}
& dq_0\left\{g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial q_0}\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi)q_0} + G(\underline{h}_0)\left[\frac{\frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi)q_0} + \frac{(1-\chi)\left(y_0 + q_0\frac{\bar{B}}{P_0}\right)}{\big[\alpha(U) - (1-\chi)q_0\big]^2}\right]\right. \\
&\qquad \left. + g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial q_0}\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} + \big[1 - G(\bar{h}_0)\big]\frac{y_0 + \frac{\alpha(D)}{\Pi R}\frac{\bar{B}}{P_0}}{\left[q_0 - \frac{\alpha(D)}{\Pi R}\right]^2}\right\} \\
&= d\alpha(D)\left\{-g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial \alpha(D)}\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi)q_0}\right. \\
&\qquad \left. - g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial \alpha(D)}\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} + \big[1 - G(\bar{h}_0)\big]\frac{y_0 + q_0\frac{\bar{B}}{P_0}}{\left[q_0 - \frac{\alpha(D)}{\Pi R}\right]^2}\frac{1}{\Pi R}\right\}
\end{aligned}$$

Since the coefficients on dq_0 and $d\alpha(D)$ are both positive, we have $\frac{\partial q_0}{\partial \alpha(D)} > 0$.

Next, we prove $\frac{\partial q_0^{\mathrm{AD}}}{\partial \alpha(D)} > 0$. Observe that, as equation (24) shows, the identity of the marginal buyer, $\bar{h}_0^{\mathrm{AD}}$, is independent of the value of $\alpha(D)$. Note also that equations (18), (19), and (20) imply that

$$q_0^{\mathrm{AD}} = \frac{1}{\Pi R}\big[\bar{h}_0^{\mathrm{AD}}\alpha(U) + (1 - \bar{h}_0^{\mathrm{AD}})\alpha(D)\big].$$

It follows that

$$\frac{\partial q_0^{\mathrm{AD}}}{\partial \alpha(D)} = \frac{1 - \bar{h}_0^{\mathrm{AD}}}{\Pi R} > 0.$$

This completes the proof.

A.3 Proof of Lemma 7

We start to prove that $\frac{\partial q_0}{\partial \chi} > 0$. To see the effect of a change in χ, let us use equations (9) and (10) to define the functions $\bar{h}_0(q_0)$ and $\underline{h}_0(q_0, \chi)$ as:

$$\bar{h}_0(q_0) \equiv \frac{1}{\alpha(U) - \alpha(D)} \big[\Pi R q_0 - \alpha(D)\big], \tag{34}$$

$$\underline{h}_0(q_0, \chi) \equiv 1 - \frac{\Pi R}{\alpha(U) - \alpha(D)} \big[\alpha(U) - (1-\chi) q_0\big]. \tag{35}$$

Note that

$$\frac{d\bar{h}_0}{dq_0} > 0, \quad \frac{\partial \underline{h}_0}{\partial q_0} > 0, \quad \frac{\partial \underline{h}_0}{\partial \chi} < 0.$$

Now we can rewrite the market clearing condition (14) as

$$\frac{\bar{B}}{P_0} + G_0\big[\underline{h}_0(q_0, \chi)\big] \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi) q_0} = \big(1 - G_0\big[\bar{h}_0(q_0)\big]\big) \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}}.$$

Differentiating this function with respect to χ and q_0, we obtain

$$\begin{aligned}
& d\chi \left\{ g(\underline{h}_0) \frac{\partial \underline{h}_0}{\partial \chi} \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi) q_0} - G_0(\underline{h}_0) \frac{q_0 \left(y_0 + q_0 \frac{\bar{B}}{P_0}\right)}{\big[\alpha(U) - (1-\chi) q_0\big]^2} \right\} \\
& \quad + dq_0 \left\{ g(\underline{h}_0) \frac{\partial \underline{h}_0}{\partial q_0} \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi) q_0} \right. \\
& \qquad \left. + G_0(\underline{h}_0) \left[\frac{\frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi) q_0} + \frac{(1-\chi)\left(y_0 + q_0 \frac{\bar{B}}{P_0}\right)}{\big(\alpha(U) - (1-\chi) q_0\big)^2} \right] \right\} \\
& = dq_0 \left\{ -g(\bar{h}_0) \frac{d\bar{h}_0}{dq_0} \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} \right. \\
& \qquad \left. + \big[1 - G_0(\bar{h}_0)\big] \left[\frac{\frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} - \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{\left(q_0 - \frac{\alpha(D)}{\Pi R}\right)^2} \right] \right\},
\end{aligned}$$

which can be rearranged as

$$\begin{aligned}
& d\chi \left\{ -g(\underline{h}_0) \frac{\partial \underline{h}_0}{\partial \chi} \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi) q_0} + G_0(\underline{h}_0) \frac{q_0 \left(y_0 + q_0 \frac{\bar{B}}{P_0}\right)}{\big[\alpha(U) - (1-\chi) q_0\big]^2} \right\} \\
& \quad = dq_0 \left\{ g(\underline{h}_0) \underline{h}_0'(q_0) \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{\alpha(U) - (1-\chi) q_0} + G_0(\underline{h}_0) \frac{\alpha(U) \frac{\bar{B}}{P_0} + (1-\chi) y_0}{\big[\alpha(U) - (1-\chi) q_0\big]^2} \right. \\
& \qquad \left. + g(\bar{h}_0) \bar{h}_0'(q_0) \frac{y_0 + q_0 \frac{\bar{B}}{P_0}}{q_0 - \frac{\alpha(D)}{\Pi R}} + \big[1 - G_0(\bar{h}_0)\big] \frac{y_0 + \frac{\alpha(D)}{\Pi R} \frac{\bar{B}}{P_0}}{\left(q_0 - \frac{\alpha(D)}{\Pi R}\right)^2} \right\}
\end{aligned}$$

Clearly, the coefficients on $d\chi$ and dq_0 are both positive, and thus $\partial q_0/\partial\chi > 0$. It then follows that $\partial\bar{h}_0/\partial\chi > 0$.

It remains to show $\partial\underline{h}_0/\partial\chi < 0$. For this, using equation (9) to define the function $q_0(\bar{h}_0)$:

$$q_0(\bar{h}_0) \equiv \frac{\alpha(U)-\alpha(D)}{\Pi R}\bar{h}_0 + \frac{\alpha(D)}{\Pi R}.$$

Thus, $dq_0/d\bar{h}_0 > 0$. Next, using equation (10), eliminate χ from the market clearing condition (14) as

$$\frac{\bar{B}}{P_0} + \frac{G_0(\underline{h}_0)}{1-\underline{h}_0}\frac{\Pi R\left[y_0 + q_0(\bar{h}_0)\frac{\bar{B}}{P_0}\right]}{\alpha(U)-\alpha(D)} = \left[1-G_0(\bar{h}_0)\right]\frac{y_0 + q_0(\bar{h}_0)\frac{\bar{B}}{P_0}}{q_0(\bar{h}_0)-\frac{\alpha(D)}{\Pi R}}.$$

Note that this equation is defined as a function of the two variables, $\bar{h}_0$ and $\underline{h}_0$ (q_0 enters as a function of $\bar{h}_0$ and all other variables are constant). Differentiating this equation with respect to $\bar{h}_0$ and $\underline{h}_0$, we obtain

$$\begin{aligned}
\frac{d\underline{h}_0}{d\bar{h}_0} = &\left(\frac{d}{d\underline{h}_0}\left[\frac{G_0(\underline{h}_0)}{1-\underline{h}_0}\right]\frac{\Pi R\left[y_0 + q_0(\bar{h}_0)\frac{\bar{B}}{P_0}\right]}{\alpha(U)-\alpha(D)}\right)^{-1} \\
&\times\left\{-\frac{G_0(\underline{h}_0)}{1-\underline{h}_0}\frac{\Pi R\frac{dq_0}{d\bar{h}_0}\frac{\bar{B}}{P_0}}{\alpha(U)-\alpha(D)} - g(\bar{h}_0)\frac{y_0 + q_0(\bar{h}_0)\frac{\bar{B}}{P_0}}{q_0(\bar{h}_0)-\frac{\alpha(D)}{\Pi R}} - \left[1-G_0(\bar{h}_0)\right]\frac{y_0 + \frac{\alpha(D)}{\Pi R}\frac{\bar{B}}{P_0}}{\left[q_0(\bar{h}_0)-\frac{\alpha(D)}{\Pi R}\right]^2}\frac{dq_0}{d\bar{h}_0}\right\} \\
&< 0.
\end{aligned}$$

Therefore, $\partial\underline{h}_0/\partial\chi < 0$. This completes the proof.

A.4 Proof of Lemma 8

As in the previous proofs we start with $\frac{\partial\rho_0}{\partial\Pi} > 0$. Now use equations (9) and (10) to define the functions $\bar{h}_0(\rho_0)$ and $\underline{h}_0(\rho_0,\Pi)$ as:

$$\bar{h}_0(\rho_0) \equiv \frac{1}{\alpha(U)-\alpha(D)}\left[\rho_0 - \alpha(D)\right], \tag{36}$$

$$\underline{h}_0(\rho_0,\Pi) \equiv 1 - \frac{1}{\alpha(U)-\alpha(D)}\left[\Pi R\alpha(U) - (1-\chi)\rho_0\right]. \tag{37}$$

Note that

$$\frac{d\bar{h}_0}{d\rho_0} > 0, \quad \frac{\partial\underline{h}_0}{\partial\rho_0} > 0, \quad \frac{\partial\underline{h}_0}{\partial\Pi} < 0.$$

Let us rewrite the market clearing condition (14) as

$$\frac{\bar{B}}{P_0} + G_0\left[\underline{h}_0(\rho_0,\Pi)\right]\frac{\Pi R y_0 + \rho_0\frac{\bar{B}}{P_0}}{\Pi R\alpha(U)-(1-\chi)\rho_0} = \left(1 - G_0\left[\bar{h}_0(\rho_0)\right]\right)\frac{\Pi R y_0 + \rho_0\frac{\bar{B}}{P_0}}{\rho_0 - \alpha(D)}.$$

Differentiating this equation with respect to Π and ρ, we get

$$
\begin{aligned}
& d\Pi\Bigg\{ g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial \Pi}\frac{\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}}{\Pi R\alpha(U)-(1-\chi)\rho_0} \\
&\qquad\qquad + G_0(\underline{h}_0)\left[\frac{Ry_0}{\Pi R\alpha(U)-(1-\chi)\rho_0}-\frac{\left(\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}\right)R\alpha(U)}{\left(\Pi R\alpha(U)-(1-\chi)\rho_0\right)^2}\right]\Bigg\} \\
&\quad + d\rho_0\Bigg\{ g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial \rho_0}\frac{\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}}{\Pi R\alpha(U)-(1-\chi)\rho_0} \\
&\qquad\qquad + G_0(\underline{h}_0)\left[\frac{\frac{\bar{B}}{P_0}}{\Pi R\alpha(U)-(1-\chi)\rho_0}+\frac{\left(\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}\right)(1-\chi)}{\left(\Pi R\alpha(U)-(1-\chi)\rho_0\right)^2}\right]\Bigg\} \\
& = d\Pi\big[1-G_0(\bar{h}_0)\big]\frac{Ry_0}{\rho_0-\alpha(D)} \\
&\quad + d\rho_0\Bigg\{-g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial \rho}\frac{\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}}{\rho_0-\alpha(D)}+\big[1-G_0(\bar{h}_0)\big]\left[\frac{\frac{\bar{B}}{P_0}}{\rho_0-\alpha(D)}-\frac{\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}}{\big[\rho_0-\alpha(D)\big]^2}\right]\Bigg\}.
\end{aligned}
$$

Rearranging this equation, we obtain

$$
\begin{aligned}
& d\Pi\Bigg\{-g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial \Pi}\frac{\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}}{\Pi R\alpha(U)-(1-\chi)\rho_0} \\
&\qquad\qquad + G_0(\underline{h}_0)\frac{Ry_0(1-\chi)\rho_0+\rho_0\frac{\bar{B}}{P_0}R\alpha(U)}{\left(\Pi R\alpha(U)-(1-\chi)\rho_0\right)^2}+\big[1-G_0(\bar{h}_0)\big]\frac{Ry_0}{\rho_0-\alpha(D)}\Bigg\} \\
& = d\rho_0\Bigg\{ g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial \rho_0}\frac{\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}}{\Pi R\alpha(U)-(1-\chi)\rho_0}+G_0(\underline{h}_0)\frac{\frac{\bar{B}}{P_0}\Pi R\alpha(U)+(1-\chi)\Pi R y_0}{\left(\Pi R\alpha(U)-(1-\chi)\rho_0\right)^2} \\
&\qquad\qquad + g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial \rho}\frac{\Pi R y_0+\rho_0\frac{\bar{B}}{P_0}}{\rho_0-\alpha(D)}+\big[1-G_0(\bar{h}_0)\big]\frac{\frac{\bar{B}}{P_0}\alpha(D)+\Pi R y_0}{\big[\rho_0-\alpha(D)\big]^2}\Bigg\}.
\end{aligned}
$$

The coefficients on $d\Pi$ and $d\rho$ are both positive. Thus, $\partial\rho_0/\partial\Pi > 0$ and $\partial\bar{h}_0/\partial\Pi > 0$.

Then note that

$$
\frac{\partial \underline{h}_0}{\partial \Pi} = -\frac{1}{\alpha(U)-\alpha(D)}\left\{R\alpha(U)-(1-\chi)\frac{\partial\rho_0}{\partial\Pi}\right\}.
$$

Hence, it suffices to show that $R\alpha(U) - (1-\chi)\frac{\partial \rho_0}{\partial \Pi} > 0$. This can be seen as:

$$
\begin{aligned}
& R\alpha(U)\left\{ g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial \rho_0}\frac{\Pi R y_0 + \rho_0 \frac{\bar{B}}{P_0}}{\Pi R\alpha(U) - (1-\chi)\rho_0} + G_0(\underline{h}_0)\frac{\frac{\bar{B}}{P_0}\Pi R\alpha(U) + (1-\chi)\Pi R y_0}{\big(\Pi R\alpha(U) - (1-\chi)\rho_0\big)^2} \right. \\
& \qquad \left. + g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial \rho}\frac{\Pi R y_0 + \rho_0\frac{\bar{B}}{P_0}}{\rho_0 - \alpha(D)} + \big[1 - G_0(\bar{h}_0)\big]\frac{\frac{\bar{B}}{P_0}\alpha(D) + \Pi R y_0}{\big[\rho_0 - \alpha(D)\big]^2} \right\} \\
& \quad - (1-\chi)\left\{ -g(\underline{h}_0)\frac{\partial \underline{h}_0}{\partial \Pi}\frac{\Pi R y_0 + \rho_0\frac{\bar{B}}{P_0}}{\Pi R\alpha(U) - (1-\chi)\rho_0} + G_0(\underline{h}_0)\frac{R y_0 (1-\chi)\rho_0 + \rho_0\frac{\bar{B}}{P_0}R\alpha(U)}{\big(\Pi R\alpha(U) - (1-\chi)\rho_0\big)^2} \right. \\
& \qquad \left. + \big[1 - G_0(\bar{h}_0)\big]\frac{R y_0}{\rho_0 - \alpha(D)} \right\} \\
& = \frac{G_0(\underline{h}_0)}{\big(\Pi R\alpha(U) - (1-\chi)\rho_0\big)^2}\left\{ \frac{\bar{B}}{P_0}\Pi R^2\alpha(U)^2 + (1-\chi)\Pi R^2\alpha(U) y_0 \right. \\
& \qquad \left. - R y_0 (1-\chi)^2\rho_0 - (1-\chi)\rho_0\frac{\bar{B}}{P_0}R\alpha(U) \right\} \\
& \quad + R\alpha(U) g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial \rho}\frac{\Pi R y_0 + \rho_0\frac{\bar{B}}{P_0}}{\rho_0 - \alpha(D)} \\
& \quad + \frac{1 - G_0(\bar{h}_0)}{\big[\rho_0 - \alpha(D)\big]^2}\left\{ R\alpha(U)\frac{\bar{B}}{P_0}\alpha(D) + \Pi R^2\alpha(U) y_0 - (1-\chi)(\rho_0 - \alpha(D)) R y_0 \right\} \\
& = \frac{G_0(\underline{h}_0)}{\big(\Pi R\alpha(U) - (1-\chi)\rho_0\big)^2}\left(\frac{\bar{B}}{P_0}R\alpha(U) + (1-\chi) R y_0 \right)\big[\Pi R\alpha(U) - (1-\chi)\rho_0\big] \\
& \quad + R\alpha(U) g(\bar{h}_0)\frac{\partial \bar{h}_0}{\partial \rho}\frac{\Pi R y_0 + \rho_0\frac{\bar{B}}{P_0}}{\rho_0 - \alpha(D)} \\
& \quad + \frac{1 - G_0(\bar{h}_0)}{\big[\rho_0 - \alpha(D)\big]^2}\left\{ R\alpha(U)\frac{\bar{B}}{P_0}\alpha(D) + (1-\chi)\alpha(D) R y_0 + R y_0\big[\Pi R\alpha(U) - (1-\chi)\rho_0\big] \right\} \\
& > 0,
\end{aligned}
$$

as desired. This completes the proof.