

# DEFAULT TRIGGERS: DOES A LIQUIDITY-BASED DEFAULT IMPLY OVER-INDEBTEDNESS AND VICE VERSA?

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## ABSTRACT

Illiquidity and over-indebtedness are common triggers of insolvency. In a discounted cash flow (DCF) framework we examine the relationship between these two triggers to verify whether these triggers are likely to coincide or whether one drives the other. We show in our analytical investigation that over-indebtedness necessarily implies danger of illiquidity at some future date. For three specific financing policies we provide sufficient and where possible, necessary conditions for the occurrence of both triggers.

**Keywords:** Valuation, Discounted Cash Flow, Insolvency, Financing Policies, Illiquidity, Default Trigger, Over-Indebtedness

## I. PROBLEM

### Two Default Triggers

There are numerous papers that deal with the valuation of firms using the discounted cash flow (DCF) method. In this context, they mainly focus on the valuation of the tax shield that emerges from interest deductibility. Today it is widely agreed that a firm's financing policy has a crucial influence on its value.<sup>1</sup> However, all financing policies have one thing in common: the value of a firm proves higher the more often the firm employs tax advantages.<sup>2</sup>

In the DCF literature it is usually assumed that the firm is able to pay off its debt in full. Another frequent assumption is that the company exists forever and never ceases economic activity. In reality, however, late payments and defaults are a frequent occurrence, if we look at the distant future.<sup>3</sup> Thus, valuation literature has been extended by incorporating default risk and today it is well known that the cost of capital declines with rising leverage due to tax advantages, but also rises due to bankruptcy costs (see e.g., Berk & DeMarzo, 2011, pp. 520-522). However, events which trigger bankruptcy are either not precisely defined or it is typically not justified why a specific trigger is applied.

Formal insolvency proceedings are regulated differently from one jurisdiction to another. However, most countries apply similar default triggers. Usually, illiquidity and over-indebtedness are typical default triggers. A company gets illiquid if its net cashflows (i.e., cash flows to equity or CFE) are negative. A company is over-indebted if the value of equity is negative (whereas market as well as book values are being used in this definition). According to German Bankruptcy Code (Insolvenzordnung), for example, a firm has to file for bankruptcy if illiquidity or over-indebtedness occurs (§ 17, § 19 InsO).<sup>4</sup> Similarly, e.g., the UK Insolvency Act initiates bankruptcy proceedings if a firm "either does not have enough assets to cover its debts (i.e., the value of assets is less than the amount of the liabilities), or it is unable to pay its debts as they fall due."

We do not investigate the influence that insolvency has on the value of a company. This has been discussed at length in the literature, however without looking at the insolvency triggers in detail. These triggers are now the focus of our research. Neither do we discuss why there are more than one insolvency trigger and why one does not suffice. We refrain from this discussion because this would require a setup with asymmetric information held by both actors, something that (to the best of our knowledge) is currently not common in the valuation literature. Here, we simply assume that there are two insolvency triggers, namely illiquidity and over-indebtedness.

Up to now the DCF literature has not looked at how these two triggers are related if we make the usual assumptions concerning the valuation of companies. In this paper we want to examine whether a company that is over-indebted will be illiquid at a future point in time and, vice versa, whether a firm that becomes illiquid will have encountered over-indebtedness. We believe that the relationship between both triggers requires more attention than they are currently given in the literature. Authors who address valuation problems seem to assume that it does not matter which insolvency trigger is used — a view we challenge. Our main focus is the relationship between these two default triggers. If there are two triggers rather than only one — how do they differ and to what extent? Are both equivalent in the sense that one implies the other under all circumstances? Should it turn out that, say, illiquidity always precedes over-indebtedness, one could claim that a DCF valuation that works with illiquidity yields a more conservative valuation than a model that uses over-indebtedness. We analytically provide evidence that over-indebtedness always implies illiquidity although the converse is not true and that the relationship between both default triggers depends on the given financing policy.

The question arises which methodology is suitable for discussing our problem. Kruschwitz and Löffler (2006) develop a simple and self-contained concept to systematically deduce valuation equations. It even allows for the incorporation of taxes on the company and on the owner level. Furthermore, various financing and distribution policies can be investigated. The concept uses a simple discrete-time calculus that is mathematically easily tractable. By using the concept of conditional expectation it is sufficient to consider only five elementary rules, and there is no need to apply the rather complicated continuous-time stochastic calculus.<sup>5</sup>

## Literature

There is a large strand of literature dealing with bankruptcy-related issues empirically. In particular, such research focuses on determinants of insolvency risk in the context of failure prediction models<sup>6</sup> (e.g., Campbell, Hilscher, & Szilagyi, 2008; Gilson, Hotchkiss, & Ruback, 2000; Laitinen & Laitinen, 2000; Mossman, Bell, Swartz, & Turtle, 1998) or on quantitative measures of direct and indirect bankruptcy costs (e.g., Altman, 1984; Andrade & Kaplan, 1998). The majority of these empirical studies has in common that it is based on a legal definition of failure, i.e., when a firm has to file for bankruptcy. This procedure has the advantage that it provides an objective criterion which allows researchers to easily identify bankrupt firms. However, it is not possible to disentangle the triggers of bankruptcy, namely insolvency and over-indebtedness.

In contrast, bankruptcy triggers are generally specified more differentiated in theoretical papers that incorporate default risk. In doing so they investigate the

impact of either over-indebtedness or illiquidity on the cost of capital, on the tax shield, and thus on firms value.

One of the first publications to focus on firm valuation with risky debt is Stiglitz (1969). In a one-period model and in the absence of taxes Stiglitz (1969) extends the results of Modigliani and Miller (1958) by incorporating risky debt. Stiglitz (1969) extensively discusses bankruptcy triggers in a multi-period context as well as the idea of illiquidity. He opts for negative (market) value of equity as an appropriate trigger for insolvency. Brennan and Schwartz (1978) analyze the influence of default risk on the value of the tax shield. They assume that all tax advantages perish in the case of bankruptcy. Both authors expect the firm to default if, on a coupon date, the value of its assets (which is in their model the value of an unlevered firm) is below some critical value. This condition is very close to the idea of Stiglitz (1969). Rapp (2006) considers insolvency in the context of DCF valuation theory. He restricts his analysis to a financing policy based on market values and explores whether the Miles-Ezzell adjustment formula remains valid. In Rapp's model, the company goes bankrupt if the value of equity is less than the value of all future payments to the creditors. Paradoxically, the model demonstrates that the firm's value increases with rising over-indebtedness.

Black and Cox (1976) as well as Leland (1994) investigate bankruptcy with safety covenants, subordination arrangements, and restrictions on the financing of interest and dividend payments. They assume the existence of a single bond with a promised final payment. If the stockholders cannot pay off the bondholders at the maturity date, default is triggered, which obviously corresponds to illiquidity. Couch, Dothan, and Wu (2012) value the tax shield of interest expenses in presence of default risk by using a barrier options methodology. In particular, default occurs if a corresponding minimum interest-coverage ratio is reached. Tham and Wonder (2001) analyze the impact of risky debt on the cost of capital. The authors use a one-period model with two states of nature to show how the relevant costs of equity and debt and the discount rate for the tax shield are determined. They also assume that bankruptcy results in a total loss of the tax shield. The default state only occurs if the firm is unable to meet its payment obligations in full. Over-indebtedness as a default trigger is ruled out by definition. Damodaran (2006) discusses several ways of incorporating the effects of financial distress into valuation theory. In particular, he provides practical guidance for estimating default probabilities and for adjusting expected cash flows as well as discount rates in context of discounted cash flow valuation. In doing so, he defines the risk of default as the likelihood of being unable to meet debt obligations. Other default triggers are not considered. Homburg, Stephan, and Weiß (2004) also focus on firm valuation in the presence of insolvency risk. With respect to financing based on market values they provide

evidence that disregarding financial distress leads to a considerable bias in firm value. Within their framework, default is triggered by an inability to pay interest and debt redemption payments in full. Again, over-indebtedness as default trigger is ruled out.

A newer paper is Koziol (2014). The author aims at providing a tractable extension of the well-known WACC approach for both default risk and bankruptcy costs. His corrected WACC is systematically higher than the usual one because first the tax component is scaled by the survivorship probability and second an extra component for bankruptcy costs is added. However, we are convinced that Koziol's proposal comprises a fatal error. Our results (in particular theorem 3.4) show that a firm with a financing policy based on market values (i.e., the most important prerequisite of the WACC approach, see for example Kruschwitz and Löffler (2006, theorem 2.9 on p. 70) will be in danger of illiquidity only if the growth rate of its cash flows is small enough. Hence, one cannot assume, as Koziol does, that "from the perspective of any state at an arbitrary date  $t$ , in which the firm is solvent, there is a unique probability  $p$  for remaining solvent until the subsequent period  $t + 1$ " (see Koziol, 2014, p. 657). Rather, that such a probability is not zero and even constant is endogenously determined by the cash flow movement itself. Apparently, Koziol (2014) tapped a trap without realizing it.

Obviously, all mentioned papers refer to one particular default trigger that can either be classified as illiquidity or as over-indebtedness. While default has been intensively investigated in prior research, until now the relationship between these two triggers has not been subject to a detailed analysis. However, for investors and financiers (e.g., in context of insolvency risk forecast) it is important to understand, whether these triggers are substitutes to each other, or whether one trigger is stricter than the other in the sense that one default criterion is met earlier. Against this background we are the first to analytically study the relationship between illiquidity and over-indebtedness within a (stochastic) DCF-framework.

Kruschwitz, Lodowicks, and Löffler (2005) is an exception in that no special default trigger is provided. Independently of a specific financing policy, Kruschwitz et al. (2005) show that a firm in danger of defaulting has exactly the same value as a firm that is not in financial distress. Since their argument relies only on the fundamental theorem of asset pricing that holds already in a world free of arbitrage, their result must also be valid in all the models mentioned above. However, it must be emphasized that the mentioned result is based on the assumption that profits due to financial recovery are taxed. This does not hold under any tax regime.

Why is the study at hand relevant? Papers that deal with the theory of business valuation, generally are based on the assumption that the managers of the business to be valued either fix its future amounts of debt or its leverage ratios. Authors

typically seem to believe that it does not matter which one of these two financing policies is postulated when it comes to the valuation of companies, which may go bankrupt at some future time. We will show in our paper, that one can not work with arbitrary assumptions here. When a financing policy with given leverage ratios is assumed, it may be that over-indebtedness can not at all occur and illiquidity is possible only under special characteristics of the firm's cash flows. Therefore, one must first think about what bankruptcy trigger one wants to use, and then think about whether this trigger is effective under the assumed financing policy.

Our paper is arranged as follows. The following section presents the model and defines the default triggers. In Section III the relationship between the two triggers under specified financing policies is analyzed. The last section summarizes.

## II. MODEL

### Notation and Assumptions

We consider firms with a lifespan of  $T$  years. However, the case of an infinite horizon,  $T \rightarrow \infty$ , is not excluded and we apply the assumption of transversality to debt as well. As far as levered firms are concerned, we use a superior  $l$  to characterize them; unlevered firms are denoted by the symbol  $u$ .

At date  $t$  the firms generate uncertain free cash flows amounting to  $\widetilde{CF}_t^l$  which are paid out to the firms' financiers (shareholders and creditors). We assume that neither the financing policy nor the eventuality of bankruptcy influences the cash flows of an unlevered firm. This is a common assumption in the DCF literature that can already be found in Modigliani and Miller (1958). The firms are subject to corporate income tax. Interest on debt  $\widetilde{I}_t$  may be deducted from the tax base.  $\widetilde{D}_t$  shall be the market value of debt raised at time  $t$ . Note, that at least future amounts of debt may be random variables. Consequently, the difference between  $\widetilde{D}_{t-1}$  and  $\widetilde{D}_t$  is the amount which the firm needs to pay back to the creditor as contracted. The actual repayment is denoted by  $\widetilde{R}_t$ ,  $\widetilde{R}_t \leq \widetilde{D}_{t-1} - \widetilde{D}_t$ . If, in the case of illiquidity, the creditors' claims cannot be paid off in full the resulting remission of debt is taxed. The tax scale  $\tau$  is deterministic and constant.

The levered firm's market value is denoted by  $\widetilde{V}_t^l$ . We evaluate the value of the company and its cash flows using the so-called risk-neutral probability measures. This risk-neutral probability differs from the individual, subjective probability measure. In order to obtain the correct value of the firm one has to discount the cash flows not with the risk-adjusted but with the riskless interest rate.<sup>7</sup> We denote the risk-neutral probability with  $Q$  and the resulting expectation of  $\widetilde{X}$  by  $E_Q[\widetilde{X}]$ . Furthermore, when evaluating expectations, the information of the investor plays

a crucial role. This information at time  $t$  is usually denoted by  $F_t$ . The conditional expectation (conditional relative to this information) is then denoted by  $E$ . Hence, the value  $\tilde{V}_t^l$  of the levered company is

$$\tilde{V}_t^l = \sum_{s=t+1}^{\infty} \frac{E_Q [ \tilde{CF}_s^l | F_t ]}{(1+r_f)^{s-t}}.$$

Because conditional expectations are random variables, the future values of the firm are random as well.

Only in  $t = 0$  does the firm define its financing policy. A financing policy is completely described by all future debt levels  $\bar{D} = [D_0, \tilde{D}_1, \dots, \tilde{D}_T]$ . It is a characteristic feature of DCF procedures that the commitment of the financing policy is made today (i.e., at  $t = 0$ ). Because many of those policies themselves depend on random variables, these future debt levels are random, too. Notice that this does not mean that the financing policy is static. Rather, the future amount of debt might be determined by events that will occur in the future. So, our financing policy is dynamic or, in a mathematically precise description, stochastic. Consider, for example, a financing policy where future debt  $\tilde{D}_t$  is a defined portion of the firm value  $\tilde{V}_t^l$ .

The creditors issue credit with a duration of one single period. At date  $t$  they charge an interest rate, which is of course known in  $t$ . In our model insolvency does not imply that the company is taken over by a liquidator. Rather positive cash flows are payments that flow from the firm to the shareholders and the tax authorities, respectively, while negative cash are payments that flow in the opposite direction without involving any insolvency trustee. Hence, in the case of negative cash flows we are rather dealing with conditions that will trigger an insolvency. This is in line with our main research question, namely that we do not scrutinize the effect of insolvency on market values but only the incidents that trigger the default and their connection.

This is the reason why we can restrict ourselves to the case of economically expected interest rates rather than having to distinguish between nominal interest and actually paid interest.<sup>8</sup> The interest rate mentioned must therefore be the riskless rate  $r_f$ . To further our calculations further, we assume that this interest rate is constant over time. This assumption is not critical, we can equip our model with an interest rate that varies over time. All results remain valid, except that the exposition gets more complicated. We therefore opt for a simpler approach.

All random variables are denoted by a tilde. The underlying state space may be infinite; a specific state is denoted by  $s$ . Notice that in a multiperiod setup a state

does not correspond to a particular point in time but rather to a path along time. If the possible states at time  $t$  are drawn from a set (say  $u, d$  as in the binomial model) then  $s$  is in fact a (possibly infinite) sequence of those elements. The set of all states is denoted by  $S$ .

We assume that the investor assigns subjective probability to the states of nature. To this end there is a probability space  $(S, F, P)$ , where  $F$  denotes a filtration of the state space and  $P$  is the subjective probability measure. The filtration  $F_t$  corresponds to the information available to the investor at time  $t$ . As usual, we assume that all variables with time-index  $t$  are  $F_t$ -measurable and hence are known to the investor at that time.

## Default Triggers

First we must define what over-indebtedness and lack of liquidity shall be in our model. The fact that both concepts are defined within a DCF setup forces us to find solutions that may be different if we were to use another framework.

We always presume a given financing policy  $\bar{D} = (D_0, \bar{D}_1, \dots)$ . From today's point of view (date 0), bankruptcy occurs at time  $t$  if the company is either illiquid or over-indebted at that time. Whereas illiquidity focuses on cash flows, we speak about over-indebtedness if the assets of the firm are lower in value than its debt.<sup>9</sup> This raises the question of how both assets and debt are measured.

11 U.S.C. § 101, 32 A (U.S. Bankruptcy Code) describes insolvency as a "financial condition such that the sum of [the] entity's debts is greater than all of [the] entity's property, at a fair valuation." Here, the book value of assets, measured at fair value (which is strictly lower than the market value<sup>10</sup>) has to be below the market value of debt. German law (German Bankruptcy Code, Insolvenzordnung) requires a filing with a bankruptcy court if one default trigger is present. As far as over-indebtedness is concerned, going concern values and realizable liquidation values may also be relevant. First, expectations about the going concern principle have to be assessed. In case of negative expectations the assets are valued according to their liquidation values; in case of positive expectations, going concern values are relevant.

In the following, however, we consider only market values instead of book values in the case of over-indebtedness. The reason for our approach is obvious: by strictly using market values we are in a position to easily produce clear conceptual relationships and provide necessary and sufficient conditions for bankruptcies. That would be much more difficult, if not impossible, if we used book values or fair values. Hence, falling back on market values is in line with the simplification strategies that prevail in economics. This forces us, however, to interpret our results reluctantly. Our considerations can therefore only be considered as a first step on the rocky path to a comprehensive analysis of different default triggers.



With the concept of illiquidity, our task seems easier. It is tempting to suggest that a company is illiquid if the owners' net cash flows turn out to be negative. But it is clearly apparent that this definition has its pitfalls when used in a DCF context. If net cash flows are positive, the company pays money to the owners. But if not, it is just the other way round: the owners pay money to the company if net cash flows appear to be negative.<sup>11</sup> Now, if a sufficient amount of money is paid to the company the firm is no longer illiquid. The owners simply rectify the unpleasant situation. Hence, if and only if the owners do not completely comply with their reserve liabilities, one can actually speak of a lack of liquidity of the company.

We hold the following: whenever the owners cannot or do not meet their funding obligations, the company effectively faces illiquidity. However, the mere existence of negative net cash flows does not automatically imply such a run of events. Hereinafter we should therefore speak of the danger of illiquidity that arises if net cash flows turn out to be negative.

**Definition 2.1 (Bankruptcy):**

For a given financing policy  $\bar{D}$ , a levered firm will be in danger of illiquidity at time  $t$  in state  $s$  if the cash flows in state  $s \in S$  do not suffice to fulfill the creditors' payment claims (interest and net redemption) at time  $t$  as contracted,

$$\tilde{CF}_t^l(s) - ((1 + r_f) \tilde{D}_{t-1}(s) - \tilde{D}_t(s)) < 0. \tag{1}$$

For a given financing policy  $\bar{D}$ , a levered firm will be over-indebted at time  $t$  in state  $s$  if the market value of debt exceeds the firm's market value,

$$\tilde{V}_t^l(s) < \tilde{D}_t(s). \tag{2}$$

Notice that both definitions refer to a future date  $t$  and the state  $s$  from today's point of view.

So far we have concentrated on levered firms. Disregarding claims of tax authorities, unlevered firms cannot go bankrupt in a legal sense. Nevertheless, it seems appropriate to apply the default triggers to unlevered firms as well. Accordingly, they reduce to  $\tilde{CF}_t^u(s) < 0$  and  $\tilde{V}_t^u(s) < 0$ . The first case gives the owners reason to reflect whether it is worth continuing to operate their business. As is reasonable, they are willing to close the cash flow gap with their personal property but only if they expect to be adequately compensated by positive cash flows in later periods. If the second condition is fulfilled, from the owner's point of view continuing to run the company would be out of the question. It would

undoubtedly be appropriate to speak of “not continuable unlevered firms.” However, for systematic reasons we use the terms “in danger of illiquidity” and “over-indebted” for unlevered firms as well, provided the mentioned formal requirements are met.

Finally, we assess the consequences of bankruptcy.<sup>12</sup> Consider a firm at date  $t = 0$  with a given financing policy  $\bar{D}$  that is in danger of illiquidity at time  $t$  in state  $s$  but not over-indebted. The management of the said firm will certainly be able to raise credit in order to ensure the continuance of the firm. If this new financing policy  $\bar{D}^*$  does not result in a lack of liquidity, the bankruptcy problem is solved. The situation could be interpreted as follows: the firm uses the new financing policy for refinancing. Illiquidity turns into a mere postponement of payments.

Yet what happens if the financing policy  $\bar{D}$  in question leads to over-indebtedness at date  $t$ ? Refinancing as in the previous paragraph is not possible since the “substance” of the firm, namely its expected future cash flows, does not suffice to satisfy the creditors’ payment claims. Moreover, we assume that credit is only granted for a single period. Therefore, the creditors anticipate at date  $t - 1$  that the loan will not be repaid in full in  $t$ . Consequently, rational creditors will not agree to issue the necessary credit in time  $t - 1$  which again has an impact on the loan granted in  $t - 2$ . As a result, we conclude that the creditors are able to detect later over-indebtedness already in  $t = 0$ . Within our framework we thus conclude that over-indebted companies are unable to realize their initial financing strategy.

### III. RELATIONSHIP BETWEEN OVER-INDEBTEDNESS AND ILLIQUIDITY

In the following we investigate whether one default trigger is more restrictive than the other, without making specific assumptions about the structure of cash flows. In the subsequent section we assume that the cash flows of the unlevered firm represent a specific stochastic process. We analyze the effects of this assumption on the relationship between the two default triggers.

#### Over-Indebtedness Implies Danger of Illiquidity

Disregarding specific assumptions concerning the dynamics of the free cash flows, we can prove that over-indebtedness implies illiquidity. This result is immediately apparent. Just realize that debts represent the present value of cash outflows while assets represent the present value of cash inflows. Having said this, it must be that at some future point of time an outflow is greater than an inflow if debts exceed assets today.

Although the result is not trivial. To this end, consider an unlevered company whose market value contains information about its future cash flows. If we know

that this market value is negative, this means that the owners of the company expect (at least in some states, not necessarily in all) that future cash flows are negative as well. That the same idea is true for levered firms is proven in the following theorem.

**Proposition 3.1** (*Over-indebtedness implies danger of illiquidity*) *If a levered company is over-indebted at time  $t$  in state  $s$ , then there is a date  $t' \geq t$  and a state  $s'$  where the firm is in danger of illiquidity.*

**Proof:**

We prove this proposition by contradiction. To this end, we consider an over-indebted firm that will never be in danger of illiquidity. In this case the inequality

$$\tilde{C}F_{t'}^l(s') - ((1 + r_f) \tilde{D}_{t'-1}(s') - \tilde{D}_{t'}(s')) \geq 0.$$

applies for all states  $s' \in S$  and times  $t' \geq t$ . Multiplying the preceding inequality with the risk-neutral probabilities and summing up leads to

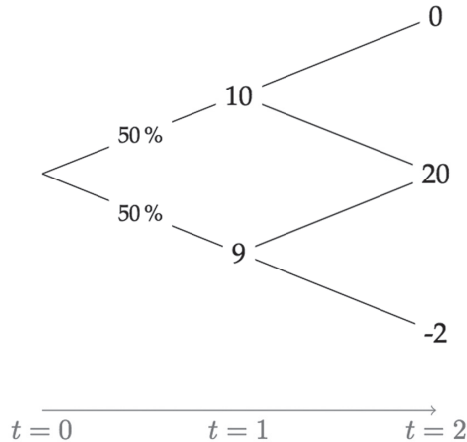
$$E_Q[\tilde{C}F_{t'}^l - ((1 + r_f) \tilde{D}_{t'-1} - \tilde{D}_{t'}) | F_t] \geq 0.$$

Dividing by  $(1 + r_f)^{t'-t}$  and adding up over all  $t$  results in

$$\begin{aligned} \sum_{t'=t+1}^T \frac{E_Q[\tilde{C}F_{t'}^l | F_t]}{(1 + r_f)^{t'-t}} &\geq \sum_{t'=t+1}^T \frac{E_Q[\left((1 + r_f) \tilde{D}_{t'-1} - \tilde{D}_{t'}\right) | F_t]}{(1 + r_f)^{t'-t}} \\ &\geq \sum_{t'=t+1}^T \frac{E_Q[r_f \tilde{D}_{t'-1} + \tilde{D}_{t'-1} - \tilde{D}_{t'} | F_t]}{(1 + r_f)^{t'-t}} \\ &\geq \sum_{t'=t+1}^T \frac{E_Q[\tilde{I}_{t'} + \tilde{R}_{t'} | F_t]}{(1 + r_f)^{t'-t}} \\ &\geq \frac{E_Q[\tilde{I}_{t+1} + \tilde{R}_{t+1} + \tilde{D}_{t+1} | F_t]}{(1 + r_f)} = \tilde{D}_t. \end{aligned}$$

Since the term on the left-hand side of the inequality is the firm value  $\tilde{V}_t^l$ , we have a contradiction to over-indebtedness. This was to be shown.

On the other hand, illiquidity does not necessarily imply over-indebtedness. We prove this statement with the help of a numerical example and assume for convenience that the firm is unlevered. Figure 1 shows the free cash flows of the company.



**Figure 1. After-Tax Cash Flows are in Danger of Illiquidity, but the Company is not Over-Indebted**

Source: This study.

The firm value is to be determined with risk-neutral probabilities, thereby assuming that an upward movement occurs with a risk-neutral probability of 50%. To simplify matters, the riskless rate amounts to zero. It turns out that the firm value is never negative, although the company generates a negative cash flow in  $t = 2$ . As one can see the firm is illiquid at  $t = 2$  (one of the cash flows is negative), but never over-indebted because the value of equity even in the down-state at  $t = 1$  is  $9 = \frac{20 - 2}{2}$  which is greater than zero. This proves the statement.

It becomes clear that every over-indebted firm will eventually be in danger of illiquidity. However, inability to pay does not necessarily result in over-indebtedness.

What is the economic meaning of those results? The first theorem states that a company that is (possibly) over-indebted at  $t$  will be in danger of illiquidity at time  $t' \geq t$ . The opposite is not necessarily true. This implies: Those who, in the interest of a cautious valuation, wish to include insolvency as early as possible should consider both insolvency triggers in their models. If they only concentrate on over-indebtedness, eventual illiquidity may take them by surprise.

### Analysis of Three Financing Policies

In the following we discuss three financing policies. First, we look at an unlevered company. Then, we consider a firm that follows an autonomous and following that, a company that follows a market value-oriented financing policy.

In the preceding section we have shown that over-indebtedness implies danger of illiquidity. The converse is not (necessarily) true. It is therefore of interest

under what circumstances we can show whether danger of illiquidity implies over-indebtedness. To this end we now concentrate on three particular financing policies. These policies will enable us not only to specify the relationship between both default triggers but also to give sufficient and necessary conditions for both triggers.

We are able to formulate such statements by assuming that the cash flows of an unlevered company are autoregressive in the sense that

$$E [\tilde{CF}_{t+1}^u \mid F_t] = (1 + g) \tilde{CF}_t^u \quad (3)$$

for a real number (growth rate)  $g > -1$ . This assumption plays an important role in the DCF literature. Without it, major results such as the stochastic version of the Gordon-Shapiro formula or the Miles-Ezzell adjustment formula cannot be proven (see e.g., Laitenberger & Löffler, 2006). Moreover, we now consider eternally active firms ( $T \rightarrow \infty$ ) with constant cost of capital  $k$ . The growth rate  $g$  is assumed to not exceed the cost of capital and to be constant. These assumptions are only made for simplicity; non-constant growth rates or non-constant cost of capital would only complicate the proofs but would not change the main results.

Our analysis provides sufficient and necessary conditions for the occurrence of both default triggers which we also illustrate by means of a numeric example. To this end we use a risk-free rate amounting to  $r_f = 10\%$ , cost of capital of  $k^{E,u} = 20\%$  and a cash flow growth rate of  $g = 5\%$ . The tax rate is set to  $\tau = 35\%$ . Furthermore, the cash flow dynamic is assumed to be based on a binomial tree, i.e., the cash flows can either grow with factor  $u$  or factor  $d$ .<sup>13</sup>

## 1. Unlevered Firms

**Proposition 3.2 (Unlevered firms)** *Assume an eternally active, unlevered firm with autoregressive cash flows, constant cost of capital and constant growth rate ( $k^{E,u} > g$ ). If the company is in danger of illiquidity at date  $t$  in some state  $s$ , it will also have a negative firm value at the same date  $t$  (over-indebtedness) for the same state  $s$ .*

This proposition bridges a gap in Kruschwitz et al. (2005), who assume that unlevered firms cannot have negative firm values; using our terminology they are assumed to be not over-indebted. It is not discussed to what condition this assumption is tied. We now find that positive cash flows of the company are sufficient and necessary for that to happen.

Notice that the market value of equity cannot be negative for corporations, since they have limited liability. The market value of non-corporations, say partnerships, can be negative though. This is the reason why we speak only of danger of insolvency instead of insolvency itself. Our theorem mainly discloses how a consistent valuation model even of an unlevered firm has to be built in order to avoid such logical contradictions.

Also, this proposition shows that for unlevered firms the result of proposition 3.1 also holds the other way round; i.e., that now over-indebtedness and danger of illiquidity are merely equivalent. In this particular case, both default triggers in fact turn out to coincide.

**Proof:**

The value of an eternally active, unlevered firm with constant cost of capital is determined by

$$\tilde{V}_t^u = \sum_{s=t+1}^{\infty} \frac{\mathbb{E}[\tilde{CF}_s^u | F_t]}{(1+k^{E,u})^{s-t}}.$$

Under the assumption of autoregressive cash flows with constant growth rate and  $k^{E,u} > g$ , the following is true (applied to state  $s$ )

$$\tilde{V}_t^u(s) = \sum_{s=t+1}^{\infty} \tilde{CF}_t^u(s) \frac{(1+g)^{s-t}}{(1+k^{E,u})^{s-t}} = \tilde{CF}_t^u(s) \frac{1+g}{k^{E,u} - g}. \quad (4)$$

It can easily be seen that the relationship between the free cash flow  $\tilde{CF}_t^u$  and the firm value  $\tilde{V}_t^u$  turns out to be deterministic. Since the factor  $\frac{1+g}{k^{E,u} - g}$  is positive by assumption, the statement of the proposition is always true.

A company that has no debt at all will be illiquid as well as over-indebted in the same instance. Therefore, in this case it is irrelevant which of the two default triggers is used.

## 2. Levered firms

We now focus on indebted firms and continue the analysis we began in proposition 3.2. We concentrate on companies that pursue either an autonomous financing policy (where any future amount of debt  $D_t$  is deterministic today) or one that is based on market values (where any future debt ratio  $\lambda_t$  is deterministic today). Furthermore, we presume the existence of an unlevered firm with identical business risk that generates autoregressive free cash flows  $\tilde{CF}_t^u$ , resulting in a firm value of  $\tilde{V}_t^u$ .

**Autonomous financing:**

We consider a levered firm that differs from the unlevered firm only in that it has a constant amount of debt  $D$ . For such a firm the following proposition shows that a sufficient large amount of debt will trigger over-indebtedness and danger of illiquidity.

**Proposition 3.3 (Autonomous financing)** *A levered firm with an infinite lifespan whose amount of debt always remains the same is over-indebted in time  $t$  in state  $s$  if and only if*

$$\tilde{C}F_t^u(s) < \frac{(1 - \tau)(k^{E,u} - g)}{1 + g} D.$$

If the condition

$$\tilde{C}F_t^u(s) < r_f(1 - \tau) D$$

is satisfied, the levered firm is in danger of illiquidity at date  $t$  in state  $s$ . Specifically, illiquidity (in state  $s$ ) implies over-indebtedness at time  $t$  (in the same state  $s$ ) if  $g < \frac{k^{E,u} - r_f}{1 + r_f}$ .

The right-hand side of the first inequality is a constant. If the cash flows of an unlevered company fall below this threshold, over-indebtedness occurs. The second inequality offers a more intuitive economic interpretation. If the cash flows go below the required interest payments after taxes, the firm is in danger of illiquidity.

In our numeric example, over-indebtedness can be avoided if the cash flows exceed 9.29% of the amount of debt. The second condition implies that cash flows are not allowed to fall below 6.5% of the amount of debt for the illiquidity risk to be ruled out. Moreover, illiquidity implies over-indebtedness, if  $g < 9.09\%$ . Under realistic assumptions this requirement is usually met.

**Proof:**

The company is in danger of over-indebtedness if  $\tilde{V}_t^l(s) < D$  holds. For the financing policy considered here,  $\tilde{V}_t^l(s) = \tilde{V}_t^u(s) + \tau D$  is valid. By plugging in, we see that

$$D > \frac{1}{1 - \tau} \tilde{V}_t^u(s)$$

is true. Plugging in Equation (4) and solving for  $\tilde{C}F_t^u(s)$  proves the proposition.

In order to prove the second statement we start with the fact that (due to constant debt) the firm is in danger of illiquidity if  $\widetilde{CF}_t^l(s) < r_f D$  holds. Making use of the relationship between the free cash flows of a levered and those of an unlevered firm

$$\widetilde{CF}_t^l(s) = \widetilde{CF}_t^u(s) + \tau r_f \widetilde{D}_{t-1} \quad (5)$$

and employing constant debt we get

$$\widetilde{CF}_t^u(s) + \tau r_f D < r_f D$$

which proves the second statement.

The relationship between both default triggers reveals that a firm can be over-indebted but able to pay at the same date  $t$  provided that the cash flows' growth rate  $g$  is sufficiently small,  $g < \frac{k^{E,u} - r_f}{1 + r_f}$ , and that the inequality holds.

$$r_f (1 - \tau) D < \widetilde{CF}_t^u(s) < \frac{(1 - \tau)(k^{E,u} - g)}{1 + g} D$$

However, this does not contradict proposition 3.1 because only one date  $t$  is considered. Moreover, it is a sufficient but not necessary condition. All over-indebted companies with an infinite lifespan and a constant amount of debt must be in danger of illiquidity sooner or later, which is what proposition 3.1 implies.

If a company is autonomously financed with constant amount of debt, then over-indebtedness comes before illiquidity. Only if the growth rates are exceptionally large this may be different.

### Financing based on market values:

Finally, we concentrate on levered firms with constant debt ratios  $\lambda = \frac{\widetilde{D}_t(s)}{\widetilde{V}_t^l(s)}$ , constant cost of capital and  $WACC > g$ .

**Proposition 3.4 (Financing based on market values)** *A levered firm with an infinite lifespan whose debt ratio always remains the same is never over-indebted. If the condition*



$$\frac{\tilde{C}F_t^u(s)}{\tilde{C}F_{t-1}^u(s)} < \frac{(1+r_f(1-\tau))(1+g)}{1 + \frac{k^{E,u} - g}{\lambda} - \frac{1+k^{E,u}}{1+r_f} r_f \tau + g}$$

is satisfied, then the firm is in danger of illiquidity at date  $t$  in state  $s$ .

Again, the implications of this proposition can be illustrated by means of a binomial tree with only the case of illiquidity being of interest. The proposition requires that the growth factor  $d$  must not fall below a specific threshold. According to the numbers in our example and the additional assumption of  $\lambda = 50\%$  the inequality is equivalent to  $d < 0.85$ .

**Proof:**

The first statement can easily be proven because for  $\lambda \leq 1$  we obtain  $\lambda \tilde{V}_t^l(s) \leq \tilde{V}_t^l(s)$  and thus  $\tilde{D}_t(s) \leq \tilde{V}_t^l(s)$ . The proof of the second statement is more sophisticated. Let us assume that the firm always remains liquid. In this case

$$\tilde{C}F_t^l(s) \geq (1+r_f) \tilde{D}_{t-1}(s) - \tilde{D}_t(s)$$

To refer to the unlevered firm's cash flows, we make use of Equation (5) and obtain

$$\tilde{C}F_t^u(s) + \tau r_f \tilde{D}_{t-1}(s) \geq (1+r_f) \tilde{D}_{t-1}(s) - \tilde{D}_t(s)$$

$$\tilde{C}F_t^u(s) \geq (1+r_f(1-\tau)) \tilde{D}_{t-1}(s) - \tilde{D}_t(s)$$

$$\tilde{C}F_t^u(s) \geq (1+r_f(1-\tau)) \lambda \tilde{V}_{t-1}^l(s) - \lambda \tilde{V}_t^l(s).$$

In the case of an eternally active firm with constant cost of capital, the firm value can be calculated by dividing the expected free cash flow by the cost of capital less the constant growth rate. For financing based on market values

$\tilde{V}_t^l(s) = \frac{(1+g)\tilde{C}F_t^u(s)}{\text{WACC} - g}$  suits. According to Miles and Ezzell (1980), for the average

cost of capital the following is true  $\text{WACC} = k^{E,u} - \frac{1+k^{E,u}}{1+r_f} r_f \tau \lambda$ .

Therefore, we obtain a total of

$$\tilde{C}F_t^u(s) \geq (1+r_f(1-\tau)) \lambda \frac{(1+g)\tilde{C}F_{t-1}^u(s)}{\text{WACC} - g} - \lambda \frac{(1+g)\tilde{C}F_t^u(s)}{\text{WACC} - g}$$

$$\left(1 + \frac{\lambda(1+g)}{\text{WACC} - g}\right) \tilde{CF}_i^u(s) \geq (1 + r_f(1 - \tau)) \lambda \frac{(1+g) \tilde{CF}_i^u(s)}{\text{WACC} - g}$$

which proves our statement.

If a company follows a market-value oriented financing policy, it will never be over-indebted. Since the use of WACC implies that the company is financed using this policy,<sup>14</sup> this result should be taken into account in any empirical research. Our model shows that WACC cannot be used as a proxy for over-indebtedness. Only illiquidity is a reasonable default trigger. Furthermore, illiquidity may only happen if the cash flows of the company do not grow enough.

## IV. SUMMARY

Schmidt (1980) advances the view that only illiquidity is a reliable bankruptcy trigger in a neoclassic model and over-indebtedness is unfeasible in such a setting.<sup>15</sup> This paper comes to a different conclusion.

We have been able to demonstrate how the two default triggers, illiquidity and over-indebtedness, are appropriately defined within the theory of discounted cash flow. We have also shown that any over-indebted company will be in danger of illiquidity sooner or later. That said, an inability to pay does not necessarily imply over-indebtedness.

Furthermore, under the additional assumption of autoregressive cash flows of the unlevered firm we have analyzed unlevered firms as well as levered companies pursuing autonomous financing or a policy based on market values. Where unlevered firms are concerned, it has turned out that both default triggers coincide.

In contrast, for autonomously financed firms with constant amount of debt, our findings suggest that both triggers can no longer be considered as substitutes. In particular, we have derived conditions for the occurrence of both, illiquidity and over-indebtedness, and provide evidence that usually illiquidity is the stricter trigger which indicates that illiquidity necessarily implies over-indebtedness at the same time, whereas over-indebted firms may at the same time still be able to pay off their debt obligations in full. Only if the growth rate of cash flows is exceptionally large, the opposite is true.

For companies that pursue a financing policy based on market values we find that over-indebtedness can never occur. A sufficient condition for the danger of illiquidity is given if growth of cash flows drops below a specific limit. This limit depends on the firm's cost of capital, tax rate, and debt ratio as well as on the riskless rate.

Our results show that it is important to clearly distinguish both default

triggers as they generally do not coincide. If one merely relies on the less strict trigger, default risk might be underestimated. Especially in the context of a conservative valuation of firms or bankruptcy risk forecasts this should be taken into account.

## NOTES

- <sup>1</sup> See Fan and So (2000) and Moosa and Li (2012) for an overview of empirical determinants of firms' capital structure.
- <sup>2</sup> In a recent study, Cheng and Tzeng (2014) provide empirical evidence that the values of leveraged firms are greater than the values of unleveraged firms if the probability of bankruptcy is not taken into account.
- <sup>3</sup> Statistics on the annual number of bankruptcy filings in Germany are published by the Federal Statistical Office. E.g., for 2013 the bankruptcy frequency amounts to 0.8%, see Statistisches Bundesamt (2013).
- <sup>4</sup> Moreover, § 18 InsO considers impending illiquidity to be a third default trigger. In our model we do not account for this case.
- <sup>5</sup> See Kruschwitz and Löffler (2006, section 1.2.2). The theory uses risk-neutral probability measures and only assumes that markets are free of arbitrage — an assumption that no serious economist would contradict. For this reason we think that for our research question, the concept developed by Kruschwitz and Löffler (2006) fits perfectly.
- <sup>6</sup> The recent study by Ni, Kwak, Cheng, and Gong (2014) provides empirical evidence on the determinants in predicting distressed firms in China. For a Korean bankruptcy prediction study see e.g., Kwak, Shi, and Kou (2012). An overview of failure prediction studies for the UK and the U.S. is provided in Charitou, Neophytou, and Charalambous (2004).
- <sup>7</sup> This procedure has been extensively and successfully used in option pricing. For the existence of the risk-neutral probability measure it is completely sufficient that the market is free of arbitrage. The assertion that any claim, complicated as it may be, is also traded (in this case the market is called complete), is not necessary to prove the fundamental theorem of asset pricing. Admittedly, in the case of an incomplete market one can show that  $Q$  does not need to be unique. But this is not a problem, at least for valuation, since every risk-neutral probability will lead to one and the same price of the firm. It is, however, a problem if the firm to be valued is not or not yet traded on the market. In this case this approach may not yield one particular value but instead a range of possible prices of the firm — and the theory developed here might not be applicable.
- <sup>8</sup> The study by Wang and Yang (2007) proposes a firm value-based bond pricing model that allows to calculate the probability of default. Within their model, default is

triggered by an inability of the issuing firm to meet its debt payment obligation.

- <sup>9</sup> Neither illiquidity nor over-indebtedness necessarily implies economically that a firm ceases to exist. The economic reason of the illiquidity procedure is rather to check whether it is advisable to continue the company or to terminate its production process.
- <sup>10</sup> Many future benefits may not be recognized as an asset in the first place and hence are not measured at fair value but at zero; e.g., intangible assets, particularly goodwill (apart from purchased goodwill).
- <sup>11</sup> Similarly e.g., Sarkar (2014) assumes that shareholders will make out-of-pocket payments in order to keep the company alive as long as the equity has some value.
- <sup>12</sup> See Leng (2013) for an empirical analysis of the bankruptcy reorganization procedure in China.
- <sup>13</sup> If one assumes equal probabilities for the occurrence of both states  $u + d = 2 \cdot (1 + 5\%)$  must hold.
- <sup>14</sup> See Kruschwitz and Löffler (2006, section 2.4.3).
- <sup>15</sup> See Schmidt (1980, p. 108).

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## **APPENDIX**

### **Executive Summary**

Discounted cash flows (DCF) is a common method of firm valuation and it is usually assumed that the firm is able to pay off its debt in full. Literature on risky debt in the context of DCF valuation is rare and focuses either on illiquidity or on over-indebtedness as a default trigger. The purpose of this paper is to demonstrate how these two triggers are appropriately defined within a DCF framework and to examine the relationship between them. In particular, we investigate to what extent both are equivalent in the sense that one implies the other under any circumstances. Our results suggest that any over-indebted company will necessarily be in danger of illiquidity at some future date but that the converse is not true. Moreover, for three specific financing policies we provide sufficient and where possible, necessary conditions for the occurrence of both triggers.

