The first order conditions for a minimum of the EGLS objective function for the original restricted VAR model are

$$\left.\frac{\partial[\mathbf{y}-(Z'\otimes I_K)R\boldsymbol{\gamma}]'(I_K\otimes\widehat{\Sigma}_u^{-1})[\mathbf{y}-(Z'\otimes I_K)R\boldsymbol{\gamma}]}{\partial\boldsymbol{\gamma}}\right|_{\widehat{\boldsymbol{\gamma}}}$$

$$= -2R'(Z\otimes I_K)(I_K\otimes\widehat{\Sigma}_u^{-1})[\mathbf{y}-(Z'\otimes I_K)R\widehat{\boldsymbol{\gamma}}] = 0.$$

Hence, $R'(Z\otimes\widehat{\Sigma}_u^{-1})\operatorname{vec}(\widehat{U})=0$. Applying the rules for the partitioned inverse (see Appendix A.10) thus gives

$$\begin{aligned}\widehat{\boldsymbol{\delta}} &= \Big(\widehat{\mathcal{U}}\widehat{\mathcal{U}}'\otimes\widehat{\Sigma}_u^{-1} \\ &\quad -(\widehat{\mathcal{U}}Z'\otimes\widehat{\Sigma}_u^{-1})R[R'(ZZ'\otimes\widehat{\Sigma}_u^{-1})R]^{-1}R'(Z\widehat{\mathcal{U}}'\otimes\widehat{\Sigma}_u^{-1})\Big)^{-1} \\ &\quad \times\operatorname{vec}(\widehat{\Sigma}_u^{-1}\widehat{U}\widehat{\mathcal{U}}').\end{aligned}$$

The usual χ^2-statistic for testing $\boldsymbol{\delta}=0$ is

$$\begin{aligned}\lambda_{LM}(h) &= \widehat{\boldsymbol{\delta}}'\Big(\widehat{\mathcal{U}}\widehat{\mathcal{U}}'\otimes\widehat{\Sigma}_u^{-1} \\ &\quad -(\widehat{\mathcal{U}}Z'\otimes\widehat{\Sigma}_u^{-1})R[R'(ZZ'\otimes\widehat{\Sigma}_u^{-1})R]^{-1}R'(Z\widehat{\mathcal{U}}'\otimes\widehat{\Sigma}_u^{-1})\Big)\widehat{\boldsymbol{\delta}}.\end{aligned}$$

Substituting the expression for $\widehat{\boldsymbol{\delta}}$, it can be seen that

$$\lambda_{LM}(h) = T\,\widehat{\mathbf{c}}_h'\widehat{\Sigma}_{\mathbf{c}}^r(h)^{-1}\widehat{\mathbf{c}}_h,$$

where

$$\begin{aligned}\widehat{\Sigma}_{\mathbf{c}}^r(h) &= \frac{1}{T}\Big(\widehat{\mathcal{U}}\widehat{\mathcal{U}}'\otimes\widehat{\Sigma}_u \\ &\quad -(\widehat{\mathcal{U}}Z'\otimes I_K)R[R'(ZZ'\otimes\widehat{\Sigma}_u^{-1})R]^{-1}R'(Z\widehat{\mathcal{U}}'\otimes I_K)\Big)\end{aligned}$$

is a consistent estimator of $\Sigma_{\mathbf{c}}^r(h)$. Thus, the situation is completely analogous to the case of an unrestricted model treated in Section 4.4.4 and we get the following result from Propositions 5.7 and C.15(5).

Proposition 5.9 (*Asymptotic Distribution of LM Statistic for Residual Autocorrelation of Restricted VAR*)
Under the conditions of Proposition 5.7,

$$\lambda_{LM}(h) \xrightarrow{d} \chi^2(hK^2).$$

■

Notice that unlike for the portmanteau test, the asymptotic distribution of the LM statistic is identical to that obtained for unrestricted VARs in Proposition 4.8. However, $\lambda_{LM}(h)$ is in general not exactly an LM statistic because the restricted estimator $\widehat{\boldsymbol{\gamma}}$ is not identical to the ML estimator. Clearly, this does not affect the asymptotic properties of the test statistic.