

As we have seen in Remark 2, the matrix operator  $\Theta(z)$  can be decomposed as

$$\Theta(z) = \Theta(1) + (1 - z)\Theta^*(z).$$

Hence, we get from (6.3.20),

$$z_t = \Theta(1)u_t + \Theta^*(L)\Delta u_t = B(1)^{-1}u_t + \Theta^*(L)\Delta u_t. \quad (6.3.21)$$

Using

$$y_t = Q^{-1}Qy_t = [\bar{\beta} : \beta_{\perp}] \begin{bmatrix} \beta' y_t \\ \bar{\beta}' y_t \end{bmatrix} = \bar{\beta}\beta' y_t + \beta_{\perp}\bar{\beta}' y_t$$

and, hence,

$$\Delta y_t = \bar{\beta}\beta' \Delta y_t + \beta_{\perp}\bar{\beta}' \Delta y_t,$$

it follows from (6.3.17) and (6.3.18) that  $\Delta y_t = z_t - \bar{\beta}\beta' z_{t-1}$ . Thus,

$$\beta_{\perp}\bar{\beta}' \Delta y_t = \beta_{\perp}\bar{\beta}' z_t.$$

Substituting the expression from (6.3.21) for  $z_t$  gives

$$\begin{aligned} \Delta y_t &= \beta_{\perp}\bar{\beta}' z_t + \bar{\beta}\beta' \Delta y_t \\ &= \beta_{\perp}\bar{\beta}' B(1)^{-1}u_t + \beta_{\perp}\bar{\beta}' \Theta^*(L)\Delta u_t + \bar{\beta}\beta' \Delta y_t := w_t. \end{aligned}$$

Solving for  $y_t = y_{t-1} + w_t$  results in

$$\begin{aligned} y_t &= y_0 + \sum_{i=1}^t w_i \\ &= y_0 + \beta_{\perp}\bar{\beta}' B(1)^{-1} \sum_{i=1}^t u_i + \beta_{\perp}\bar{\beta}' \Theta^*(L) \sum_{i=1}^t \Delta u_i + \bar{\beta}\beta' \sum_{i=1}^t \Delta y_i \\ &= y_0 + \beta_{\perp}\bar{\beta}' B(1)^{-1} \sum_{i=1}^t u_i + \beta_{\perp}\bar{\beta}' \Theta^*(L)(u_t - u_0) + \bar{\beta}\beta'(y_t - y_0) \\ &= \beta_{\perp}\bar{\beta}' B(1)^{-1} \sum_{i=1}^t u_i + \beta_{\perp}\bar{\beta}' \Theta^*(L)u_t + \bar{\beta}\beta'y_t + y_0^*, \end{aligned} \quad (6.3.22)$$

where  $y_0^* := y_0 - \beta_{\perp}\bar{\beta}' \Theta^*(L)u_0 - \bar{\beta}\beta'y_0$ . Using  $\beta'y_t = \beta'z_t$ , the term  $\bar{\beta}\beta'y_t = \bar{\beta}\beta'z_t$  is seen to have a representation

$$\bar{\beta}\beta'z_t = \bar{\beta}\beta'\Theta(L)u_t$$

and, thus,  $\beta_{\perp}\bar{\beta}' \Theta^*(L)u_t + \bar{\beta}\beta'y_t$  has an MA representation

$$\Xi^*(L)u_t = [\beta_{\perp}\bar{\beta}' \Theta^*(L) + \bar{\beta}\beta'\Theta(L)]u_t.$$