The lemma implies the following limiting result for the LS estimator $\widehat{\mathbf{\Pi}}$.

Result 1
Let

$$D = \begin{bmatrix} T^{1/2} I_r & 0 \\ 0 & T I_{K-r} \end{bmatrix}.$$

Then

$$\begin{aligned} &\operatorname{vec}[Q(\widehat{\mathbf{\Pi}} - \mathbf{\Pi})Q^{-1}D] \\ &\xrightarrow{d} \begin{bmatrix} \mathcal{N}(0, (\Gamma_z^{(1)})^{-1} \otimes \Sigma_v) \\ \operatorname{vec}\left\{ \Sigma_v^{1/2} \left(\int_0^1 \mathbf{W}_K d\mathbf{W}_K' \right)' \Sigma_v^{1/2} \begin{bmatrix} 0 \\ I_{K-r} \end{bmatrix} \right. \\ \left. \times \left([0 : I_{K-r}] \Sigma_v^{1/2} \left(\int_0^1 \mathbf{W}_K \mathbf{W}_K' ds \right) \Sigma_v^{1/2} \begin{bmatrix} 0 \\ I_{K-r} \end{bmatrix} \right)^{-1} \right\} \end{bmatrix}. \end{aligned} \tag{7.1.6}$$

■

Proof:

$$\begin{aligned} &Q(\widehat{\mathbf{\Pi}} - \mathbf{\Pi})Q^{-1}D \\ &= \left[T^{-1/2} \sum_{t=1}^T v_t z_{t-1}^{(1)\prime} : T^{-1} \sum_{t=1}^T v_t z_{t-1}^{(2)\prime} \right] \\ &\quad \times D \begin{bmatrix} \sum_t z_{t-1}^{(1)} z_{t-1}^{(1)\prime} & \sum_t z_{t-1}^{(1)} z_{t-1}^{(2)\prime} \\ \sum_t z_{t-1}^{(2)} z_{t-1}^{(1)\prime} & \sum_t z_{t-1}^{(2)} z_{t-1}^{(2)\prime} \end{bmatrix}^{-1} D \\ &= \left[\left(T^{-1/2} \sum_{t=1}^T v_t z_{t-1}^{(1)\prime} \right) \left(T^{-1} \sum_{t=1}^T z_{t-1}^{(1)} z_{t-1}^{(1)\prime} \right)^{-1} \right. \\ &\quad \left. : \left(T^{-1} \sum_{t=1}^T v_t z_{t-1}^{(2)\prime} \right) \left(T^{-2} \sum_{t=1}^T z_{t-1}^{(2)} z_{t-1}^{(2)\prime} \right)^{-1} \right] + o_p(1). \end{aligned}$$

The last equality follows from Lemma 7.1(4). The result in (7.1.6) is obtained by vectorizing this matrix and applying Lemma 7.1(2), (3), and (5) and the continuous mapping theorem (see Appendix C.8). ■

An immediate implication of Result 1 follows.