Result 2
The estimator $\widehat{\mathbf{\Pi}}$ is asymptotically normal,

$$\sqrt{T}\text{vec}(\widehat{\mathbf{\Pi}} - \mathbf{\Pi}) \xrightarrow{d} \mathcal{N}\left(0, \beta(\Gamma_z^{(1)})^{-1}\beta' \otimes \Sigma_u\right), \tag{7.1.7}$$

and $\beta(\Gamma_z^{(1)})^{-1}\beta'$ can be estimated consistently by

$$\left(T^{-1}\sum_{t=1}^{T} y_{t-1}y_{t-1}'\right)^{-1}.$$

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Proof:

$$\begin{aligned}
&\sqrt{T}Q(\widehat{\mathbf{\Pi}} - \mathbf{\Pi})Q^{-1} \\
&= Q(\widehat{\mathbf{\Pi}} - \mathbf{\Pi})Q^{-1}D\begin{bmatrix} I_r & 0 \\ 0 & T^{-1/2}I_{K-r}\end{bmatrix} \\
&= \left[\left(T^{-1/2}\sum_{t=1}^{T} v_t z_{t-1}^{(1)\prime}\right)\left(T^{-1}\sum_{t=1}^{T} z_{t-1}^{(1)} z_{t-1}^{(1)\prime}\right)^{-1}\right. \\
&\qquad \left. : T^{-1/2}\left(T^{-1}\sum_{t=1}^{T} v_t z_{t-1}^{(2)\prime}\right)\left(T^{-2}\sum_{t=1}^{T} z_{t-1}^{(2)} z_{t-1}^{(2)\prime}\right)^{-1}\right] + o_p(1)
\end{aligned}$$

from the proof of Result 1 and, hence,

$$\begin{aligned}
&\sqrt{T}\text{vec}[Q(\widehat{\mathbf{\Pi}} - \mathbf{\Pi})Q^{-1}] = (Q^{-1\prime} \otimes Q)\sqrt{T}\text{vec}(\widehat{\mathbf{\Pi}} - \mathbf{\Pi}) \\
&\xrightarrow{d} \begin{bmatrix} \mathcal{N}\left(0, (\Gamma_z^{(1)})^{-1} \otimes \Sigma_v\right) \\ 0 \end{bmatrix}.
\end{aligned}$$

Premultiplying by $Q' \otimes Q^{-1}$ and recalling the definition of Q, gives a multivariate normal limiting distribution with covariance matrix

$$(Q' \otimes Q^{-1})\left(\begin{bmatrix} (\Gamma_z^{(1)})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \otimes \Sigma_v\right)(Q \otimes Q^{-1\prime})$$

or

$$[\beta : \alpha_\perp]\begin{bmatrix} (\Gamma_z^{(1)})^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \beta' \\ \alpha_\perp' \end{bmatrix} \otimes Q^{-1}\Sigma_v Q^{-1\prime}$$

which implies (7.1.7) because $\Sigma_v = Q\Sigma_u Q'$.

Now consider

$$\left(T^{-1}\sum_{t=1}^{T} y_{t-1}y_{t-1}'\right)^{-1} = Q'\left(T^{-1}\sum_{t=1}^{T} z_{t-1}z_{t-1}'\right)^{-1} Q$$