and

$\nu_1, \dots, \nu_K$ are the corresponding orthonormal eigenvectors.

The log-likelihood function in (7.2.19) is maximized for

$$\begin{aligned}
\beta' &= \widetilde{\beta}' := [\nu_1, \dots, \nu_r]' S_{11}^{-1/2}, \\
\alpha &= \widetilde{\alpha} := \Delta Y M Y_{-1}' \widetilde{\beta} \left(\widetilde{\beta}' Y_{-1} M Y_{-1}' \widetilde{\beta} \right)^{-1} = S_{01} \widetilde{\beta} (\widetilde{\beta}' S_{11} \widetilde{\beta})^{-1}, \\
\mathbf{\Gamma} &= \widetilde{\mathbf{\Gamma}} := (\Delta Y - \widetilde{\alpha} \widetilde{\beta}' Y_{-1}) \Delta X' (\Delta X \Delta X')^{-1}, \\
\Sigma_u &= \widetilde{\Sigma}_u := (\Delta Y - \widetilde{\alpha} \widetilde{\beta}' Y_{-1} - \widetilde{\mathbf{\Gamma}} \Delta X)(\Delta Y - \widetilde{\alpha} \widetilde{\beta}' Y_{-1} - \widetilde{\mathbf{\Gamma}} \Delta X)'/T.
\end{aligned}$$

The maximum is

$$\max \ln l = -\frac{KT}{2} \ln 2\pi - \frac{T}{2} \left[\ln |S_{00}| + \sum_{i=1}^{r} \ln(1 - \lambda_i) \right] - \frac{KT}{2}. \qquad (7.2.20)$$

■

Proof: From Chapter 3, Section 3.4, it is known that for any fixed α and β the maximum of $\ln l$ is attained for

$$\widetilde{\mathbf{\Gamma}}(\alpha\beta') = (\Delta Y - \alpha\beta' Y_{-1}) \Delta X' (\Delta X \Delta X')^{-1}.$$

Thus, we replace $\mathbf{\Gamma}$ in (7.2.19) by $\widetilde{\mathbf{\Gamma}}(\alpha\beta')$ and get the concentrated log-likelihood

$$\begin{aligned}
&-\frac{KT}{2} \ln 2\pi - \frac{T}{2} \ln |\Sigma_u| \\
&\quad -\frac{1}{2} \mathrm{tr} \left[(\Delta Y M - \alpha\beta' Y_{-1} M)' \Sigma_u^{-1} (\Delta Y M - \alpha\beta' Y_{-1} M) \right].
\end{aligned}$$

Hence, we just have to maximize this expression with respect to α, β, and Σ_u. We also know from Chapter 3 that, for given α and β, the maximum is attained if

$$\widetilde{\Sigma}(\alpha\beta') = (\Delta Y M - \alpha\beta' Y_{-1} M)(\Delta Y M - \alpha\beta' Y_{-1} M)'/T$$

is substituted for Σ_u. Consequently, we have to maximize

$$-\frac{T}{2} \ln |(\Delta Y M - \alpha\beta' Y_{-1} M)(\Delta Y M - \alpha\beta' Y_{-1} M)'/T|$$

or, equivalently, minimize the determinant with respect to α and β. Thus, all results of Proposition 7.3 follow from Proposition A.7 of Appendix A.14. ■

The solutions $\widetilde{\beta}$ and $\widetilde{\alpha}$ of the optimization problem given in the proposition are not unique because, for any nonsingular $(r \times r)$ matrix Q, $\widetilde{\alpha} Q^{-1}$