where x_t is the stochastic part which is assumed to have a VECM representation without deterministic terms and μ_t is the deterministic term, as in Chapter 6, Section 6.4. We will start with the easiest although most unrealistic case where no deterministic term is present and, thus, $\mu_t = 0$. Most of the discussion will focus on likelihood ratio (LR) tests and close relatives of them because they are very common in applied work and they also fit well into the present framework. Some comments on other procedures will be provided in Section 8.2.9.

8.2.1 A VECM without Deterministic Terms

Based on Proposition 7.3, it is easy to derive the likelihood ratio statistic for testing a specific cointegration rank $r = r_0$ of a VECM against a larger rank of cointegration, say $r = r_1$. Consider the VECM without deterministic terms,

$$\Delta y_t = \mathbf{\Pi} y_{t-1} + \mathbf{\Gamma}_1 \Delta y_{t-1} + \cdots + \mathbf{\Gamma}_{p-1} \Delta y_{t-p+1} + u_t, \tag{8.2.1}$$

where y_t is a process of dimension K, $\mathrm{rk}(\mathbf{\Pi}) = r$ with $0 \leq r \leq K$, the $\mathbf{\Gamma}_j$'s ($j = 1, \dots, p-1$) are $(K \times K)$ parameter matrices and $u_t \sim \mathcal{N}(0, \Sigma_u)$ is Gaussian white noise, as in Chapter 7, Section 7.2.3. For simplicity we assume that the process starts at time $t = 1$ with zero initial values (i.e., $y_t = 0$ for $t \leq 0$). Alternatively, the initial values may be any fixed values.

Suppose we wish to test

$$H_0 : \mathrm{rk}(\mathbf{\Pi}) = r_0 \quad \text{against} \quad H_1 : r_0 < \mathrm{rk}(\mathbf{\Pi}) \leq r_1. \tag{8.2.2}$$

Under normality assumptions, the maximum of the likelihood function for a model with cointegration rank r is given in Proposition 7.3. From that result, the LR statistic for testing (8.2.2) is seen to be

$$\begin{aligned} \lambda_{LR}(r_0, r_1) &= 2[\ln l(r_1) - \ln l(r_0)] \\ &= T\left[-\sum_{i=1}^{r_1} \ln(1-\lambda_i) + \sum_{i=1}^{r_0} \ln(1-\lambda_i)\right] \\ &= -T \sum_{i=r_0+1}^{r_1} \ln(1-\lambda_i), \end{aligned} \tag{8.2.3}$$

where $l(r_i)$ denotes the maximum of the Gaussian likelihood function for cointegration rank r_i. Obviously, the test value is quite easy to compute, using the eigenvalues from Proposition 7.3.

It turns out, however, that the asymptotic distribution of the LR statistic under the null hypothesis for given r_0 and r_1 is nonstandard. In particular, it is not a χ^2-distribution. It depends on the number of common trends $K - r_0$ under H_0 and on the alternative hypothesis. Two different pairs of hypotheses have received prime attention in the related literature:

$$H_0 : \mathrm{rk}(\mathbf{\Pi}) = r_0 \quad \text{versus} \quad H_1 : r_0 < \mathrm{rk}(\mathbf{\Pi}) \leq K \tag{8.2.4}$$