Valuing the Tax Shield under Asymmetric Taxation

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Abstract

Models on enterprise valuation show that the effective or marginal tax rate (*ETR*) increases linearly with the debt ratio, implying that tax benefits from debt are very important. Empirical research has repeatedly emphasized that this result cannot be sustained, and that tax benefits are in reality much less relevant to valuation. It is an open question whether this impression can also be maintained within a theoretical model.

All theoretical models so far assume that the tax rate is constant and identical for gains and losses. In this paper, we attempt to analytically determine the value of a tax shield assuming that gains and losses are taxed differently. We want to precisely determine the impact of a non-constant tax rate on the value of the tax shield. Previous research could only integrate this asymmetry by employing empirical methods and simulation studies, as an analytical solution had not yet been presented.

Looking at a very popular financing policy we are able to present a closed-form solution for the effective tax rate. Our results reveal that this value, instead of being a linear function of the debt ratio, is rather concave, sustaining repeated empirical observations. However, our results also show that the "power" or "strength" of this concavity is not enough to explain the empirical results concerning the impact of the tax shield. Therefore, adding an asymmetric taxation is not enough to determine the empirically observed puzzle of tax shield valuation.

Keywords

■

JEL-Classification

■

1. Introduction

How important are the tax benefits from debt? Using formal models, this question was raised and answered as early as Modigliani and Miller (1963) using a very simple financing policy (constant debt). Later Miles and Ezzell (1980) were able to give a closed-form solution for another financing policy (constant leverage ratio) that remains one of the most popular assumptions in finance today. Then, research moved to empirical and simulation studies.

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The literature has in particular looked at the ratio of tax shield value and enterprise value of a levered firm to epitomize the influence of a corporate tax on firm value. This ratio sometimes is called "marginal tax rate" (also MTR) in the empirical literature. Using this term in our model would be rather confusing, given that the tax shield is a tax benefit and given that we analyze the entire tax benefit and not a marginal surplus. To be precise, we will analyze a coefficient that measures the present value of all future tax shields relative to the present value of all the future income, given that the company is levered (see Eq. 4 below). Hence, we will denote it as "effective tax shield ratio" *ETR*.

Looking at the theoretical results above, we identify a common element. In Modigliani-Miller's as well as Miles–Ezzell's case the *ETR* is linear in today's debt ratio (see below). If we use the concept of elasticity the immediate result is that the tax benefit has an elasticity of one with respect to the debt ratio. This implies that taxes are very important when valuing companies and such results should be empirically observed, in particular when tax rates are changing.

And this is where the issue gets interesting. Many papers have repeatedly argued that the effect of debt on the value of the tax shield is much less than both theories – Modigliani-Miller or Miles–Ezzell – predict. Myers et al. (1998) have argued that taxes are of third-order importance in the hierarchy of corporate decisions, hence much less than the model's presage.

Many reasons can be mentioned (financial flexibility, use of non-debt tax shields, pecking-order theory, target ratings of rating agencies, costs of financial distress), but one idea immediately comes to mind: Until now, in any analytical model where corporate taxes are introduced, gains and losses are treated symmetrically. But the treatment of gains and losses differs considerably across national tax code regulations and is usually not symmetric. Many countries grant loss carry-back and offer schemes for loss carry-forward.[1] These accumulated tax losses can be quite enormous, as Sarkar (2014) has emphasized.[2] Losses and non-deducted interest can be restricted by the amount of tax-deductible losses to a certain proportion of current-year profits or may be ultimately lost because a substantial amount of shares of a loss making firm is transferred to a new owner.[3]

If, for example, losses cannot be imputed at all but gains are subject to tax this will have an impact on the value of the tax shield and hence also the elasticity. We would expect that the influence is of an order less than one. Up to now this result could only be verified using simulation models or empirical studies; particularly worth mentioning are Shevlin (1990), Graham (1996a, 1996b, 2000, 2003, 2006), Graham and Mills (2008), Graham and Kim (2009), Blouin et al. (2010). The effect of a different taxation of gains and losses (so-called "tax convexity") has been analyzed, for example, in Sarkar (2008) using a continuous-time setup. Sarkar was only able to simulate first results. Koch (2013, Part E) thoroughly discussed the weaknesses of such simulation studies and why we cannot relay on simulations alone: Koch shows that the estimation of marginal tax rates using a random walk approach involves a huge measurement error. With simulations, one must rely on few numerical values to deduce structural statements for all possible numbers.

This is the point where our paper continues. Our aim is to present an analytical model where gains are taxed differently than losses, in one case even presenting a closed-form solution for the value of the tax shield. Our approach clearly shows that the elasticity of the *ETR* with respect to the debt ratio is less than one, pointing in the right direction. In our examples, the value of the tax shield turns out to be a concave function of the leverage ratio.

We have already mentioned the consensus that the value of the tax shield is of less importance for corporate decisions. It will turn out that theoretical results will support this concord. Although the *ETR* is not linear in the debt ratio, substantially lowering the tax shield compared to the symmetric case turns out to be challenging. For this to be the case rather unrealistic assumptions are required: For example, the cost of capital would have to be very far away from the riskless rate. In the end, our paper shows that (as in the empirical and simulation literature) formal models so far cannot provide a convincing answer why tax shields are so low.

2. Assumptions

We assume a market with the usual properties. There is a risk-free asset with interest rate r_f which, for simplicity, is assumed to be constant over time. The market is free of arbitrage and hence there is a risk-neutral probability measure Q such that any claim can be evaluated using the discounted Q -expected cash flow of that claim.[4]

The firm we want to consider has unlevered pre-tax cash flows CF_t^u that are auto-regressive,

$$CF_t^u = CF_{t-1}^u(1 + \varepsilon_t) \tag{1}$$

for all $t > 0$.[5] The random variables ε_t are assumed to be independent and identically distributed (iid), with the expectation of zero and satisfy $\varepsilon_t > -1$.

Given these assumptions, the price V_t^u of an unlevered (post-tax) cash flow stream CF_s^u ($s = t + 1, \ldots$) with a tax rate τ is given by the sum of its Q -expected and discounted value:

$$V_0^u = \sum_{t=1}^{\infty} \frac{E_Q[(1-\tau)\cdot CF_t^u]}{(1+r_f)^t}. \tag{2}$$

τ is a firm income tax rate, cash flows (instead of accounting incomes) are subject to taxation. We ignore personal income taxes in order to keep our model tractable.[6]

The unlevered company has after-tax cash flows of $(1-\tau)CF_t^u$. The levered company can deduct taxes if there are no losses. Hence, its after-tax cash flow is [7]

$$CF_t^l = CF_t^u - \tau\left(CF_t^u - r_f D_{t-1}\right)^+.$$

This gives a tax shield at time t of

$$\begin{aligned} TS_t :\ &= CF_t^u - \tau\left(CF_t^u - r_f D_{t-1}\right)^+ - (1-\tau)CF_t^u \\ &= \begin{cases} \tau r_f D_{t-1} & \text{if } CF_t^u > r_f D_{t-1} \\ \tau CF_t^u & \text{else.} \end{cases} \\ &= \tau \min(CF_t^u, r_f D_{t-1}). \end{aligned} \tag{3}$$

In our analysis we observe cases where the cash flows can take values smaller than the required interest payments, which would usually mean a default of the corporation. In order to be able to work with such cash flows we follow the assumptions in Kruschwitz and Löffler (2006, Chap. 2.2.4). There, it is shown that default (under very mild assumptions) does not change the valuation equations.

Lastly, we assume that the capital costs of the unlevered firm are constant over time.[8] From this, for the unlevered company we immediately obtain

$$V_t^u = \frac{(1-\tau)\, CF_t^u}{k}.$$

Introducing debt, the now levered company will use an amount of debt at time t. An equation applies to the valuation of this company, which is quite similar to Eq. (2). However, its value V_0^l will be determined by the cash flows of the levered firm. Before we focus on two different types of financing policies that play an important role in the theory of business valuation a general result is almost self-evident: The value of today's tax shield is concave given any future debt level D_t.[9] This result will now be amplified using the following two financing policies:

~~Fixed leverage ratios~~

Fixed leverage ratios: The first financing policy is characterized by the fact that company management fixes deterministic leverage ratios l_t for the future. This is well known in the literature as it is the prerequisite for using *WACC* in firm valuation, see Miles and Ezzell (1980, p. 722). Because the future values of the indebted firm V_t^l are stochastic, the same applies for the future amounts of debt, $l_t\, V_t^l = D_t$. For simplicity, assume that the future leverage ratio is constant over time, $l_t = l_0\ (\forall t > 0)$.

~~Fixed amounts of debt~~

Fixed amounts of debt: The second financing policy has management fixing the future amounts of debt, D_t, deterministically. For convenience, assume that this amount remains constant over time, $D_t = D_0\ (\forall t > 0)$. Modigliani and Miller (1963) discussed this type of policy. As the future values of the indebted firm are stochastic, then the future debt ratios of the firm must also be stochastic under this financing policy, $l_t = D_0/V_t^l$.

ETR is finally being defined by

$$ETR := 1 - \frac{V_0^u}{V_0^l}. \quad \text{4(4)}$$

We have already mentioned that different definitions have been suggested in the literature to capture the effect of asymmetric taxation. Shevlin (1990) as well as Graham (2003) consider a "corporate marginal tax rate … defined as the change in the present value of the cash flow paid to (or recovered from) the tax authorities as a result of earning one extra dollar of TI [tax income] in the current period" (see Shevlin 1990, p. 1). Analogously, Graham and Kim (2009) write that the "ETR [marginal tax] rate measures the present value tax consequences of earning an extra dollar of income today" (p. 416). As can be seen our definition is in line with these descriptions.

We are interested in closed-form solutions for the ETR, particularly if gains and losses are taxed differently.

3. Main Results

3.1. Financing Policy with Constant Leverage Ratios

First assume that firm management follows a financing policy with a deterministic and constant leverage ratio, $l_0 = l_1 = \ldots = l$. This case was addressed by Miles and Ezzell (1980). The result is

$$\left(1 - \frac{1+k}{1+r_f}\,\frac{r_f}{k}\,\tau\, l\right) V_0^l = V_0^u,$$

where k is the cost of capital of the unlevered company. It is straightforward to determine the ETR if gains and losses are taxed symmetrically:

$$ETR_{\text{symmetric}} = \frac{1+k}{1+r_f}\,\frac{r_f}{k}\,\tau\, l. \quad \text{5(5)}$$

The derivation of a closed-form equation for the ETR under asymmetric taxation is harder. Assume that losses cannot be imputed at all. Let $WACC$ represent the weighted average cost of capital and $WACC = (1-\tau)\, CF_t^u / V_t^l$ for some t . We get the following result.[10]

Proposition 1 (Asymmetric Taxation of Gains and Losses) If gains are taxed, while losses are not, then WACC is not stochastic and even constant. Furthermore,

$$ETR_{\text{asymmetric}} = \frac{1+k}{1+r_f}\frac{r_f}{k}\,\tau\, l f\left(\frac{r_f\, l\,(1-\tau)}{WACC}\right), \qquad (6)$$

where $f(\cdot)$ is a monotonically decreasing function with values between 0 and 1 .

Comparing equations (5) and (6) with each other reveals an interesting fact. The *ETR* differ from each other only by the factor of $f\left(\frac{r_f\, l\,(1-\tau)}{WACC}\right)$ and $0 \leq f\left(\frac{r_f\, l\,(1-\tau)}{WACC}\right) \leq 1$ must hold.

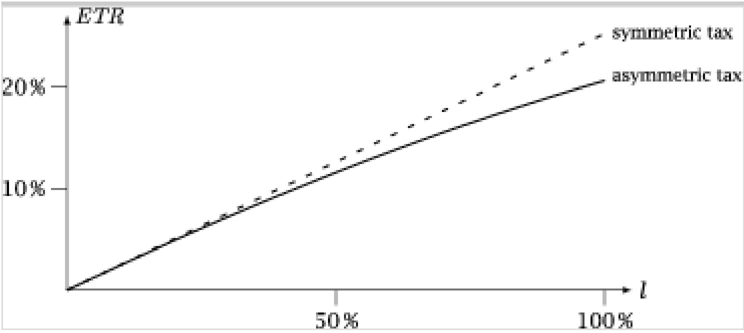
Consider an example. Assume that ε_t regarding Q is uniformly distributed on the interval $[-1, 1]$; calculating the function $f(\cdot)$ for this case yields[11]

$$f(x) = 1 - \frac{x}{4}, \qquad x \in (0, 1].$$

Fig. 1 shows the functional relationship between the *ETR* and the leverage ratio, its main influencing factor. It is easy to see that *ETR* under symmetrical taxation is a linear function of the debt ratio, while it is a concave function under asymmetric taxation.

Fig. 1

ETR under constant leverage ratios ($k = 6\,\%$, $r_f = 5\,\%$, $\tau = 30\,\%$) with ε_t regarding Q being uniformly distributed on $[-1, 1]$. The dotted line gives the *ETR* when gains and losses are taxed symmetrically, the straight lines shows the *ETR* if losses cannot be imputed



Nevertheless, we motivated our approach with the empirical observation that the value of the tax shields does not seem to be very important when valuing companies. Introducing asymmetric taxation does not provide a conclusive answer to that puzzle. Even our example shows that producing significantly smaller tax shield with asymmetric taxation compared to the symmetric case comes at the cost of unrealistic assumptions. If, in our example, the cost of capital k is increased to a reasonable level $ETR_{\text{asymmetric}} \approx ETR_{\text{symmetric}}$ results. This example supports the claim that also in a theoretical model an asymmetric taxation is not the sufficient answer why taxes are of less importance in corporate decisions.

The example already shows that asymmetric taxation hardly explains why “typical conditions” usually result in a rather small tax shield. Looking at the argument of the function $f(\cdot)$ carefully proves to be helpful. It can easily be seen that $\frac{r_f\, l\,(1-\tau)}{WACC}$ turns out to be quite a small number and for small x the function $f(x)$ approaches 1 . This result can formally be stated as follows:[12]

Proposition 2 If

$$\frac{r_f}{WACC}\; l\,(1-\tau) \le \inf_t \frac{CF^u_{t+1}}{CF^u_t}$$

applies, then $f\left(\frac{r_f\, l\,(1-\tau)}{WACC}\right) = 1$ *, and asymmetric taxation cannot affect the value of the tax shield.*

If the growth rate of cash flows is not negative the right hand side approaches one. With negative rates it may be less than that. But it should not decrease too much under normal conditions. On the left hand, however, we have an expression which, under ordinary circumstances, is clearly less than one and often very small. Hence, Proposition 2 will regularly hold.[13]

3.2. Financing Policy with Constant Amounts of Debt

Now assume that the firm follows a financing policy with deterministic and constant amounts of debt, $D_0 = D_1 = \ldots = D$. The future values of the levered firm are stochastic. Hence, due to $l_t := D/V^l_t$ the future leverage ratios are stochastic as well. By contrast, the previous leverage ratio was a number at any future time t .

Under symmetric taxation the value of the levered firm at each time is

$$V^l_t = V^u_t + \tau D\,,$$

From this, immediately

$$\frac{V_t^l - V_t^u}{V_t^l} = \frac{\tau D}{V_t^l} = \tau\, l_t \,.$$

These terms are stochastic for any $t > 0$. Only the current *ETR* (i.e., at $t = 0$) is deterministic. Under symmetric taxation of gains and losses the *ETR* at time $t = 0$ is deterministic and is described as

$$ETR_{\text{symmetric}} = \tau\, l_0 \,. \qquad 7(7)$$

The result is different if gains and losses are taxed differently.[14]

Proposition 3 (Asymmetric Taxation of Gains and Losses) *If gains are taxed, while losses are not imputed at all, then the ETR at time $t = 0$ is deterministic. Depending on the debt, ETR attains a value between τl_0 and $\frac{\tau}{1+\tau}$; the greater the amount of debt, the greater the ETR.*

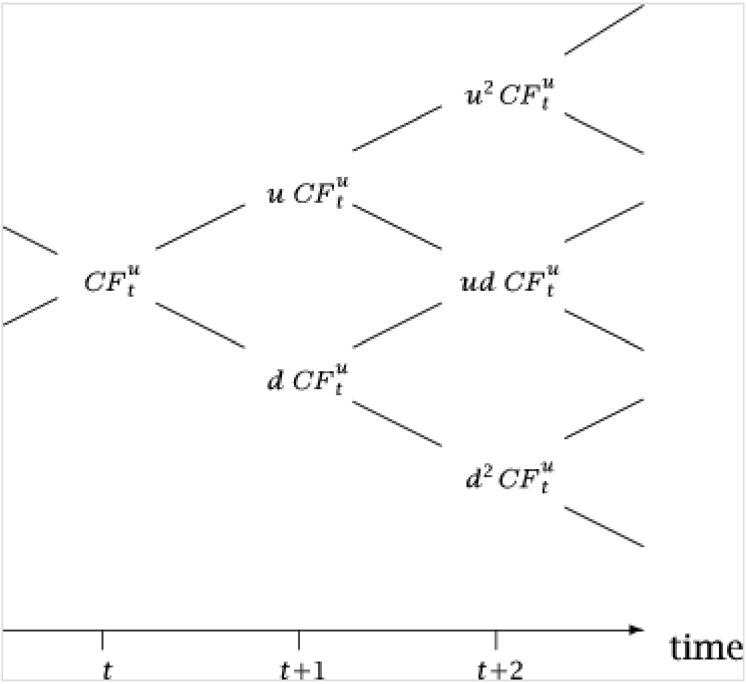
The first value τl_0 materializes if D is sufficiently small. The second value $\frac{\tau}{1+\tau}$ results if $l_0 \geq \frac{1}{1+\tau}$. Unfortunately, we do not arrive at a closed-form solution for the *ETR* if D yields results that are located between τl_0 and $\frac{\tau}{1+\tau}$.

Asymmetric taxation is, of course, without any meaning if $CF_t^u \geq r_f D_t$ holds. Obviously, symmetric and asymmetric taxation result in the same tax shields.[15]

Consider an example. The cash flows of the company follow a binomial tree as shown in Fig. 2, with the initial value $CF_0^u = 1$ and the growth factors $u = 1.0$ and $d = 0.9$. The cost of capital is given by $k = 5\%$, the risk-free rate is $r_f = 3\%$, and the tax rate is $\tau = 60\%$. Then the risk-neutral probabilities can be determined using option pricing theory.[16]

Fig. 2

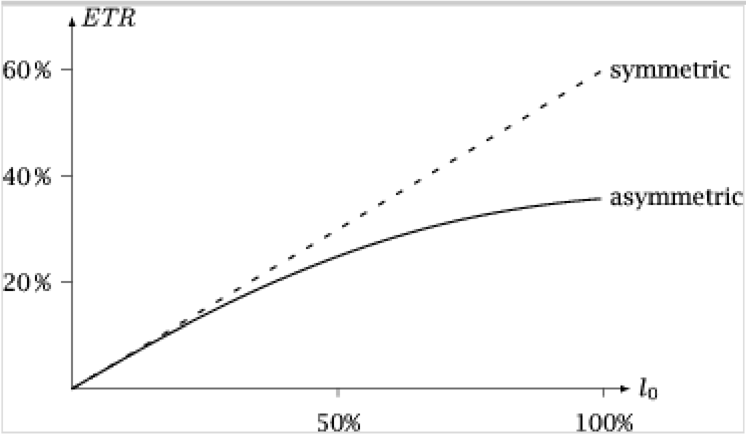
Binomial tree of cash flows in the second example



Calculating the *ETR* as a function of today's leverage ratio l_0 gives the result shown in Fig. 3. It is clear that beyond a certain amount of debt the *ETR* no longer increases, because the resulting losses can no longer be offset against taxes. Again we see that *ETR* is a concave function of debt under asymmetric taxation and a linear function under symmetric taxation. This agrees with empirical results.

Fig. 3

ETR under constant amounts of debt ($k = 5\,\%$, $r_f = 3\,\%$, $\tau = 60\,\%$, $CF_0^u = 1$), when cash flows follow a binomial tree as in ~~figure~~Fig. 2 with $u = 1.0$ and $d = 0.9$. The dotted line gives the *ETR* when gains and losses are taxed symmetrically, the straight lines shows the *ETR* if losses cannot be imputed



4. Conclusion

Evaluating a firm requires a lot of information, including the value of the firm's *ETR*. In the past 25 years, there have been articles on the estimation of *ETR* under asymmetric taxation (gains are taxed, losses are tax-free), but all papers published so far work with empirical methods and simulation studies. This paper is the first to attempt an analytical determination of the *ETR* and our findings for two popular financing policies are summarized in Table 1.

Table 1

~~Effective tax shield ratio *ETR* under symmetric and asymmetric taxation~~

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Financing policy	Losses are taxed	Losses cannot be imputed
Fixed leverage ratios	$\frac{1+k}{1+r_f}\frac{r_f}{k}\tau l$	$\frac{1+k}{1+r_f}\frac{r_f}{k}\tau l f\left(\frac{r_f\, l\,(1-\tau)}{WACC}\right)$
Fixed amounts of debt	τl_0	$\tau l_0 \rightarrow \frac{\tau}{1+\tau}$

Empirical studies that recognize a different tax treatment of gains and losses indicate that taxes are less important in firm evaluation than previously implied by theoretical papers. Although in our examples the differences between the symmetric and the asymmetric tax shield valuation can be substantial, with a constant leverage ratio in realistic cases these differences turn out to be low. And we assumed that losses and non-deducted interest are ultimately lost and can be carried neither forward nor backward – in the latter case timing effects come into play that will reduce the differences further. Our paper supports the claim that asymmetric taxation is not the major reason why taxes are of less importance in business decisions at least in the case of a constant debt ratio. This has not been shown in analytical models so far.

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5. Appendix

5.1. Proof of Proposition 1

From Eq. (3), using Eq. (2), the value of the levered company is

$$V_t^l = \sum_{s=t+1}^{\infty} \frac{E_Q\left[(1-\tau)CF_s^u + \tau\min(CF_s^u, r_f l V_{s-1}^l)|\mathcal{F}_t\right]}{(1+r_f)^{s-t}}$$

or by employing the stochastic and time-dependent variable

$$WACC_s := \frac{(1-\tau)CF_s^u}{V_s^l}$$ A1(A1)

$$V_t^l = \sum_{s=t+1}^{\infty} \frac{E_Q\left[(1-\tau)CF_s^u + \tau\min(CF_s^u, \frac{r_f\, l\,(1-\tau)}{WACC_{s-1}} CF_{s-1}^u)|\mathcal{F}_t\right]}{(1+r_f)^{s-t}}$$

It follows from Kruschwitz and Löffler ~~(2015), Proposition 2~~(2015, Proposition 2) that there must be a unique solution. However, it is not obvious how to determine that solution. Claiming that

$$WACC = \frac{(1-\tau)CF_s^u}{V_s^l}$$ A2(A2)

is deterministic and constant will prove to be correct. From our first assumption we get

$$CF_s^u = CF_{s-1}^u(1+\varepsilon_s)$$

for an iid variable ε_s . Using Eq. (A2), insertion yields

$$\begin{aligned} V_t^l &= V_t^u + \sum_{s=t+1}^{\infty} \frac{E_Q\left[\tau\min(CF_{s-1}^u(1+\varepsilon_s), \frac{r_f\, l\,(1-\tau)}{WACC}\, CF_{s-1}^u) \,\middle|\, \mathcal{F}_t\right]}{(1+r_f)^{s-t}} \\ &= V_t^u + \tau\sum_{s=t+1}^{\infty} \frac{E_Q\left[CF_{s-1}^u \min\left(1+\varepsilon_s, \frac{r_f\, l\,(1-\tau)}{WACC}\right) \,\middle|\, \mathcal{F}_t\right]}{(1+r_f)^{s-t}}. \end{aligned}$$ ~~10~~(A3)

The random variables

$$CF_{s-1}^u = CF_0^u(1+\varepsilon_1)(1+\varepsilon_2)\cdots(1+\varepsilon_{s-1})$$

and

$$\min\left(1+\varepsilon_s, \frac{r_f\, l\,(1-\tau)}{WACC}\right)$$

are independent of each other. Under this condition, the expectation of the product equals the product of the expectations. Hence using $x := (r_f\, l\,(1-\tau))/WACC$ yields

~~11~~(A4)

$$V_t^l = V_t^u + \tau \sum_{s=t+1}^{\infty} \frac{E_Q\left[CF_{s-1}^u \mid \mathcal{F}_t\right] E_Q\left[\min\left(1+\varepsilon_s, \frac{r_f\, l\,(1-\tau)}{WACC}\right) | \mathcal{F}_t\right]}{(1+r_f)^{s-t}}$$
$$= V_t^u + \tau \sum_{s=t+1}^{\infty} \frac{r_f\, l}{WACC} \frac{E_Q\left[(1-\tau)CF_{s-1}^u | \mathcal{F}_t\right] E_Q\left[\min\left(\frac{1+\varepsilon_s}{x}, 1\right) | \mathcal{F}_t\right]}{(1+r_f)^{s-t}} .$$

We now focus on a function

$$f(x) =_{\text{Def}} E_Q\left[\min\left(\frac{1+\varepsilon_t}{x}, 1\right) | \mathcal{F}_s\right], \qquad t > s$$

for $x > 0$. This function is dependent on three terms, namely x , the information $\mathcal{F}_s$, and the random variable ε_t . The latter being iid, this is an unconditional expectation that depends only on x . Therefore

$$f(x) = E_Q\left[\min\left(\frac{1+\varepsilon_t}{x}, 1\right)\right]$$

must hold. Now it can easily be shown that when x is small,

$$\lim_{x\to 0} f(x) = E_Q\left[\min\left(\lim_{x\to 0} \frac{1+\varepsilon_t}{x}, 1\right)\right] = 1 ,$$

because $\varepsilon_t > -1$, and when x is large

$$\lim_{x\to \infty} f(x) = E_Q\left[\min\left(\lim_{x\to \infty} \frac{1+\varepsilon_t}{x}, 1\right)\right] = 0 .$$

The function is monotonically decreasing with x .

We can now determine the tax shield using the newly defined function $f\left(\frac{r_f\, l\,(1-\tau)}{WACC}\right) = 1$. Inserting the term into Eq. (~~11~~A4) yields

$$V_t^l = V_t^u + \frac{r_f\, l}{WACC} \tau \sum_{s=t+1}^{\infty} \frac{E_Q\left[(1-\tau)CF_{s-1}^u | \mathcal{F}_t\right] E_Q\left[\min\left(\frac{(1+\varepsilon_s)\ WACC}{r_f\, l\,(1-\tau)}, 1\right)\right]}{(1+r_f)^{s-t}}$$
$$= V_t^u + \frac{r_f\, l}{WACC} \tau f\left(\frac{r_f\, l\,(1-\tau)}{WACC}\right) \sum_{s=t+1}^{\infty} \frac{E_Q\left[(1-\tau)CF_{s-1}^u | \mathcal{F}_t\right]}{(1+r_f)^{s-t}}$$
$$= V_t^u + \frac{r_f\, l}{WACC} \tau f\left(\frac{r_f\, l\,(1-\tau)}{WACC}\right) \frac{(1-\tau)CF_t^u + V_t^u}{1+r_f} . \qquad \text{~~12~~(A5)}$$

This is a closed-form equation for the tax shield.

This result is based on the mere assumption of *WACC* being deterministic and constant. If we can trust this result, our assumption was justified. We have to show that if there is a constant and deterministic $WACC$, there is a unique solution. To this end, insert the capital costs equations into Eq. (~~12~~A5):

$$\frac{(1-\tau)\,CF_t^u}{WACC} = \frac{(1-\tau)\,CF_t^u}{k} + \frac{r_f\,l}{WACC\,(1+r_f)}\,\tau f\left(\frac{r_f\,l\,(1-\tau)}{WACC}\right)\left(1+\frac{1}{k}\right)(1-\tau)\,CF$$

This can easily be transformed to

$$WACC = k - \frac{1+k}{1+r_f} r_f\,\tau\,l f\left(\frac{r_f\,l\,(1-\tau)}{WACC}\right).$$

This corresponds to the adjustment formula of Miles and Ezzell (1980) except for the term $f(\cdot)$.

To assure ourselves that a unique solution exists for $WACC$, consider two cases. For $WACC \to 0$ the left-hand side (LHS) of the equation goes to zero, while the right-hand side (RHS) goes to $k > 0$. So the RHS is larger than the LHS. Assuming, however, that $WACC \to \infty$, the LHS goes beyond all limits and is positive, while the RHS remains finite. Because of the monotonicity of the function there can be only one unique solution for $f(\cdot)$. The ETR results easily from Eq. (~~12~~A5):

$$\begin{aligned} ETR_{\text{asymmetric}} &= 1 - \frac{\frac{CF_t^u}{k}}{\frac{CF_t^u}{WACC}} = 1 - \frac{WACC}{k} \\ &= \frac{1+k}{1+r_f}\frac{r_f}{k}\,\tau\,l f\left(\frac{r_f\,l\,(1-\tau)}{WACC}\right). \end{aligned}$$

This completes the proof.

5.2. The example with constant leverage ratio and proof of Proposition 2

The function $f(\cdot)$ from

$$f(x) = E_Q\left[\min\left(\frac{1+\varepsilon_t}{x}, 1\right)\right],$$

can be determined if ε is regarding Q uniformly distributed on $[-a, a]$. Then

$$f(x) = \int_{-a}^{a} \min\left(\frac{1+\varepsilon}{x}, 1\right)\frac{1}{2a}\,d\varepsilon\ .$$

We have, since $x > 0$

$$\frac{1+\varepsilon}{x} \geq 1 \iff x-1 \leq \varepsilon.$$

Distinguish two cases. First, if $x-1 \leq -a$ then the integral is given by

$$f(x) = \int_{-a}^{a} 1\frac{1}{2a}\,d\varepsilon = 1.$$

The second case corresponds to $-a \leq x-1 \leq a$. Then, the integral is given by

$$f(x) = \int_{-a}^{x-1} \min\left(\frac{1+\varepsilon}{x}, 1\right)\frac{1}{2a}\,d\varepsilon + \int_{x-1}^{a} \min\left(\frac{1+\varepsilon}{x}, 1\right)\frac{1}{2a}\,d\varepsilon.$$

This can be simplified

$$\begin{aligned} f(x) &= \int_{-a}^{x-1} \frac{1+\varepsilon}{x}\frac{1}{2a}\,d\varepsilon + \int_{x-1}^{a} 1\cdot\frac{1}{2a}\,d\varepsilon \\ &= \frac{1}{2} + \frac{1}{2a} - \frac{1}{4a}\left(x + \frac{(1-a)^2}{x}\right). \end{aligned}$$

Finally, Proposition 2 must be proved. From the assumptions we have

$$\frac{r_f}{WACC}\,l\,(1-\tau) \leq \inf_t \frac{CF^u_{t+1}}{CF^u_t} \implies \frac{r_f}{WACC}\,l\,(1-\tau) \leq (1+\varepsilon_t).$$

With $x = \frac{r_f}{WACC}\,l\,(1-\tau)$ we get

$$1 \leq \frac{1+\varepsilon_t}{x},$$

and the integral is given by

$$f(x) = E_Q\left[\min\left(\frac{1+\varepsilon_t}{x}, 1\right)\right] = E_Q[1] = 1.$$

This was to be shown.

5.3. Proof of Proposition 3

Recall Eq. (3)

$$TS_t = \tau \min(CF^u_t, r_f D_{t-1}).$$

From this, for the levered firm with constant amounts of debt

$$V_0^l = V_0^u + \tau \sum_{t=1}^{\infty} \frac{E_Q[\min(CF_t^u, r_f D)]}{(1+r_f)^t}.$$

Obviously, we must now distinguish two cases. If $r_f D \leq CF_t^u$ ("sufficiently small amount of debt"), it is the known case

$$V_0^l = V_0^u + \tau D$$

and therefore, as with symmetric taxation

$$ETR_{\text{asymmetric}}^{\text{case 1}} = \tau\, l_0. \qquad \text{~~A6~~(A6)}$$

However, if $r_f D > CF_t^u$ ("sufficiently large amount of debt"), then

$$V_0^l = V_0^u + \tau \sum_{t=1}^{\infty} \frac{E_Q[CF_t^u]}{(1+r_f)^t} = (1+\tau)V_0^u$$

applies. From this follows directly

$$ETR_{\text{asymmetric}}^{\text{case 2}} = \frac{\tau}{1+\tau}. \qquad \text{~~A7~~(A7)}$$

Note that $l_0 \geq 0$ must be provided. Hence, for sufficiently small D the *ETR* may be vanishingly small, but can never become negative. For sufficiently large debt, the *ETR* is positive and independent of the extent of debt. As a result, in the general case (where $\min(CF^u{}_t, r_f D)$ applies) we realize that

$$ETR_{\text{asymmetric}} \leq \tau \min\left(l_0, \frac{1}{1+\tau}\right) \qquad \text{~~A8~~(A8)}$$

must hold.

Furthermore, *ETR* is a monotonically increasing function in D , starting at $D = 0$ where $ETR = \tau l_0$. Consequently, for increasing D , the effective tax shield ratio must grow from τl_0 to $\frac{\tau}{1+\tau}$.

This completes the proof.

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[1] Loss carry-back is granted, for example, in the United States, France, Germany, United Kingdom, and Japan. The carry-back volume is unlimited with the exception of Germany and carry-back periods range from 1 to 3 years. Many countries offer schemes for loss carry-forward. Germany, again, limits the carry-forward volume. Periods, in which tax losses carried forward are valid, range from 5 years to infinity. See Dwenger and Walch (2014) or Canefield (1999).

[2] Waegenaere et al. (2003) discuss the valuation of carry-forward losses.

[3] Currently, six of the G20-countries (Brazil, Germany, France, Italy, Japan, Saudi Arabia) apply such regulations to restrict loss carry-forward provisions by a certain proportion of profit. In the US, loss carry-backs are similarly restricted. Losses are ultimately lost in Germany if (within five years) more than one half of the equity is transfered to a new entity (§8c Abs. 1 Satz 1–4 KStG). In the US section 382 of the IRC limits the use of the tax loss carryover of a corporation that is acquired in a merger or stock purchase. The annual limitation is the product of the value of the acquired corporation and the long-term tax-exempt interest rate, see IRC § 382(b)(1).

[4] The existence of this risk-neutral measure is called the fundamental theorem and has been used extensively in option pricing.

[5] This assumption is now standard in the valuation literature, for formal details of this approach see Kruschwitz and Löffler (2006 , Chap. 1). Notice that our assumption (1) is slightly different from the literature (multiplicative instead of additive noise). This is due to the fact that only with multiplicative noise terms a meaningful valuation result can be established, see the discussion of transversality in Kruschwitz and Löffler (2015 , Sects. 3.1 and 3.2).

[6] Kruschwitz and Löffler (2006 , Chap. 3) and in particular Sect. 3.2 discuss the problems with personal income taxes and valuation.

[7] The symbol X^+ means $\max(X, 0)$.

[8] The relation of a constant or even deterministic cost of capital to the assumptions cited above is not straightforward. See Kruschwitz and Löffler (2006 , Chap. 1) for details.

[9] Notice that D_t is a random variable. Therefore, a general definition of concavity has to be applied. However, even with this definition the function $\min(x, \cdot)$ is well known to be concave for any x . The expectation is then the sum of concave functions with a positive scalar and is also concave. For details see Hiriart-Urruty and Lemarechal (2001 , Chap. B.1, Proposition 2.1.1).

[10] We have moved the proof to the Appendix.

[11] For a calculation of the function $f(\cdot)$ see the Appendix.

[12] See Appendix for evidence.

[13] We would like to thank a reviewer for pointing us to that fact.

[14] Again, the proof is in the Appendix.

[15] See (A6) in the Appendix.

[16] See Kruschwitz and Löffler (2006 , p. 42f.).